# Generalized Norm-compatible Systems on Unitary Shimura Varieties

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par

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"Si l'ordre est le plaisir de la raison, le désordre est le délice de l'imagination."

—Paul Claudel, Le Soulier de Satin ou Le pire n'est pas toujours sûr[1], 1929

[1] Théâtre, II, éd. Gallimard, coll. «Bibliothèque de la Pléiade», 1965, p. 952.

To my late mother and my beloved $\sim$.

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R. B.

# ABSTRACT


We define and study in terms of integral Iwahori–Hecke algebras a new class of geometric operators acting on the Bruhat-Tits building of connected reductive groups over $p$-adic fields. These operators, which we call $\mathbb{U}$-operators, generalize the geometric notion of "successors" for trees with a marked end. The first main contributions of the thesis are:

(i) the integrality of the $\mathbb{U}$-operator over the spherical Hecke algebra using the compatibility between Bernstein and Satake homomorphisms,

(ii) in the unramified case, the $\mathbb{U}$-operator attached to a cocharacter is a right root of the corresponding Hecke polynomial.

In the second part of the thesis, we study some arithmetic aspects of special cycles on (products of) unitary Shimura varieties, these cycles are expected to yield new results towards the Bloch–Beilinson conjectures. As a global application of (ii), we obtain:

(iii) the horizontal norm relations for these GGP cycles for arbitrary $n$, at primes where the unitary group splits.

The general local theory developed in the first part of the thesis, has the potential to result in a number of global applications along the lines of (iii) (involving other Shimura varieties and also vertical norm relations) and offers new insights into topics such as the Blasius–Rogawski conjecture as well.

# Contents

# CHAPTER I

# INTRODUCTION

## I.1 Introduction

### I.1.1 Birch and Swinnerton–Dyer conjecture for elliptic curves

Elliptic curves and Galois representations have received a significant attention in the past few decades for several reasons. First and foremost, Andrew Wiles' proof of Fermat's last theorem was based on the modularity theorem for elliptic curves over $\mathbb{Q}$ [Wil95, BCDT01].

In addition, the Birch and Swinnerton–Dyer conjecture (the BSD conjecture) for elliptic curves still remains one of the most fundamental open problems in modern number theory and arithmetic algebraic geometry. The conjecture predicts that the rank of the group of rational points on an elliptic curve should be the same as the order of vanishing of its $L$-function at $s = 1$. Morally, it formalizes the philosophy that the more points one has on an elliptic curve over a number field, the more points one gets (on average) over the residue fields at the different finite places. There is also a refined conjecture that expresses the leading term of the Taylor expansion of the $L$-function in terms of local and global invariants for the elliptic curve.

The strongest evidence towards this open problem is obtained for curves with analytical rank at most one: the rank part of the BSD holds and the Tate–Shafarevitch group is finite. This is a consequence of[1]

(i) Kolyvagin's Heegner points Euler system, the central ingredient for bounding the size of the corresponding Selmer group [Kol90, Gro91].

(ii) The Gross–Zagier formula relating the derivative of the corresponding $L$-function at the central point to the height of the Heegner point [GZ86].

The BSD conjecture generalizes in different directions to closely related mathematical objects. For instance:

• There is a version for abelian varieties over number fields as well as for modular forms of arbitrary weight. More generally, one can formulate a precise analogue for Galois representations via Selmer groups; the latter are subgroups of the first Galois cohomology group of the representation determined by specified local conditions.

• One has the Bloch–Kato–Beilinson conjectures [BK90]. These, may be seen as a remarkable attempt to unify BSD with a few other conjectures that were wandering in number theory's paysage for a few decades, such as the main conjecture of Iwasawa theory. More precisely, the Bloch–Kato–Beilinson conjectures are a sequence of statements that generalize BSD to higher-dimensional objects[2], and thus, relate it to the other major fundamental research program in number theory - the Langlands program.

### I.1.2 Anti-cyclotomic main conjecture of Perrin-Riou

A related problem in the theory of Galois representations is the study of various Selmer groups over $\mathbb{Z}_p$-extensions of number fields (the Iwasawa theory of Galois representations). Since Selmer groups behave nicely over such infinite extensions, in the basic case of elliptic curves, one typically obtains strong Iwasawa-theoretic results unconditionally on the rank of the elliptic curve [PR95, Kat04, How04, BD05].

[1] together with some analytic results due to Murty-Murty and Bump-Friedberg-Hoffstein [BFH90, MM97].

[2] The BSD Conjecture in its original form is a statement about modular and Shimura curves.

### I.1.3 Euler systems

After the seminal work of Kolyvagin [Kol90], it was clear that "Euler systems"[3] would play a central role to make progress on the open problems discussed above. More accurately, they currently appear to be our most efficient tool to approach cases of Bloch–Kato type conjectures[4].

Unfortunately, these objects are rare in the mathematical literature, and the list of constructed ones was rather short [Kol90, Kat04]. But in recent years, there has been a surge of new candidates proposed by various people (Cornut [Cor18], Jetchev [Jet16], Boumasmoud–Brooks–Jetchev [BBJ18, BBJ16], Loeffler–Zerbes–Lei [LLZ14, LLZ17], Loeffler–Zerbes–Kings [KLZ15, KLZ17] and, Loeffler–Zerbes–Skinner [LSZ17]). Judging by the success of their predecessors, one expects that these new Euler Systems should soon yield significant new developments in arithmetic geometry.

Euler systems are delicate rigid objects in the following sense: they [5] are collections of *compatible* Galois cohomology classes in $\mathbf{H}^1(L\,,T)$ indexed by fields $L$, ($F \subset L \subset F[\infty]$), for some fixed infinite abelian extension $F[\infty]$ of a number field $F$ where $T$ is a $p$-adic representation of $\mathrm{Gal}(F^{ab}/F)$. These (Galois) cohomology classes are constrained by a set of relations, of which there are three major types: the horizontal (or tame) relations, the vertical relations[6] and the congruence relations.

Establishing these relations for the early Euler systems constructed out of Heegner points or Siegel units is an ad-hoc exercise in CM theory and modular functions. Yet, the situation changed dramatically for the new applicants to the point that most of the difficulty for obtaining constructions in higher dimensions seems to be now concentrated in this new bottleneck.

As of now, there is no general strategy for establishing the desired distribution relations for a given collection of special cycles on a Shimura variety. This thesis attempts to contribute to this part of the theory by proposing a general construction: our central tool is a class

[3] They allow to model analytic objects (local L-functions) in terms of cohomological data and thus, relate the analytic and arithmetic side of the BSD conjetcure.

[4] There is also the method of Ribet(–Mazur–Wiles), generalized to the $\mathbf{GL}(2)$-setting by Skinner and Urban, and the method of Wiles ("$R = T$") that yields results for $L$-functions adjoint motives.

[5] I would like to avoid here the difficult task of defining what an "Euler/Kolyvagin" system is.

[6] the vertical relations are only required to expand the reach of the Euler System in Iwasawesque directions.

of operators acting on Bruhat–Tits buildings of reductive groups that we call $\mathbb{U}$-operators. *They bridge the actions of the Galois group and the Hecke algebra on the free abelian group generated by these cycles.*

### I.1.4 From classical Heegner points towards a general construction

In order to outline the ideas behind our general methods, we apply them to the classical case of a modular curve, by placing the classical approach in a more conceptual group theoretic framework.

Let $N$ be an integer and $E/\mathbb{Q}$ be an imaginary quadratic field with ring of integers $\mathcal{O}_E$. Assume that all primes of $N$ split in $E$ and let $\mathcal{N}$ be an ideal of $\mathcal{O}_E$ of norm $N$. If $m$ is prime to $N$, the isogeny $\mathbb{C}/\mathcal{O}_m \to \mathbb{C}/(\mathcal{N} \cap \mathcal{O}_m)^{-1}$ corresponds to a Heegner point $x_m$ in $X_0(N)(E[m])$, where $E[m]$ denotes the ring class field of conductor $m$ and $\mathcal{O}_m = \mathbb{Z} + m\mathcal{O}_E$ is the corresponding order of $E$. Set $\mathrm{CM}_E := \{x_m \colon m \text{ prime to } N\}$. The points in $\mathrm{CM}_E$ are related by the following norm-compatibilities [Dar04, Proposition 3.10]:

**Proposition** I.1.4.1 (***Distribution relations***). *Let $m$ be an integer and $\ell$ a prime which is unramified over $E$. We also suppose that $m\ell$ is prime to $N$. In this case, we have*

*(i)* ***Tame relations:*** *Let $\lambda$ be a prime of $E$ that lies over $\ell$. If $\ell \nmid m$, then*

$$\mathrm{Tr}_{E[m\ell]/E[m]}\, x_{m\ell} = \left(T_\ell - \mathrm{Fr}_\lambda - \left(\frac{\mathfrak{d}_E}{\ell}\right) \mathrm{Fr}_\lambda^{-1}\right) x_m, \tag{I.1}$$

*where, $T_\ell$ denotes the Hecke operator corresponding to*

$$[\mathbf{GL}_2(\mathbb{Z}_\ell)\, diag(\ell, 1)\, \mathbf{GL}_2(\mathbb{Z}_\ell)],$$

*and $\mathrm{Fr}_\lambda \in Gal(E[\infty]/E)$ denotes the geomteric Frobenius.*

*(ii)* ***Vertical relations:*** *if $\ell \mid m$, then*

$$\mathrm{Tr}_{E[m\ell]/E[m]}\, x_{m\ell} = T_\ell x_m - x_{m/\ell}. \tag{I.2}$$

Now, consider the Hecke polynomial

$$H_\ell(X) = X^2 - T_\ell X + \ell S_\ell,$$

where, $S_\ell$ denotes the Hecke operator corresponding to the double coset

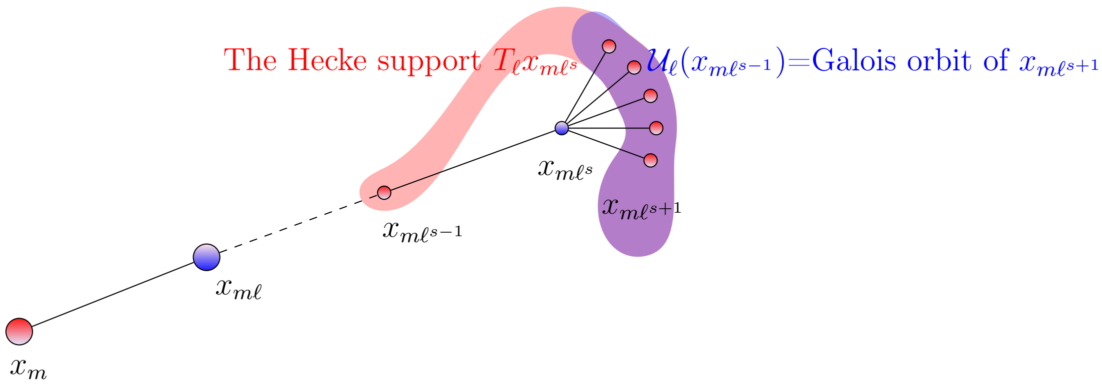
$$[\mathbf{GL}_2(\mathbb{Z}_\ell)\mathrm{diag}(\ell,\ell)\,\mathbf{GL}_2(\mathbb{Z}_\ell)].$$

There exists an operator $\mathcal{U}_\ell \in \mathrm{End}_{\mathbb{Z}}\mathbb{Z}[\mathrm{CM}_E]$, a variant of the combinatorial "successor" operator[7], verifying the following properties:

1. **The Hecke side:**
$$H_\ell(\mathcal{U}_\ell) = 0 \text{ in } \mathrm{End}_{\mathbb{Z}}\mathbb{Z}[\mathrm{CM}_E]. \tag{I.3}$$

2. **The Galois side:** Let $\ell \nmid m$. For $s \geq 1$, we have
$$\mathrm{Tr}_{E[m\ell^{s+1}]/E[m\ell^s]}\, x_{m\ell^{s+1}} = \mathcal{U}_\ell x_{m\ell^s}, \tag{I.4}$$

3. **The Congruence side:** If $\ell \nmid m$, then
$$(\mathcal{U}_\ell - \mathrm{Fr}_\lambda)x_m \equiv 0 \mod \left(\ell - \left(\frac{\mathfrak{d}_E}{\ell}\right)\right)\mathbb{Z}[\mathrm{CM}_E], \tag{I.5}$$
where $\mathfrak{d}_E$ denotes the different ideal of $E$.




It is an easy exercise to verify that the *"Hecke side (I.3) and Galois side (I.4)"* implies the vertical distribution relations (I.2), while the *"Hecke side (I.3) and Congruence side (I.5)"* implies the tame distribution relation (I.1).

[7]It may be compared with Cornut–Vatsal operator $T_P^u$ in [CV07, 6.3].

## I.2 Main results of the thesis

### I.2.1 Previous work

In a series of papers, Jetchev and Boumasmoud–Brooks–Jetchev constructed a novel Euler system using higher dimensional CM 1-cycles on certain 3-dimensional unitary Shimura varieties for $\mathbf{U}(2,1) \times \mathbf{U}(1,1)$ associated to a CM-extension $E/F$. This family of cycles is expected to yield new results towards the Bloch–Beilinson conjectures.

The work of Jetchev [Jet16] introduced these special cycles and proved formulae for their local fields of definition at primes that are inert in $E/F$, together with tame distribution relations comparing cycles ramified and unramified at such primes. In [BBJ16] we extend these two results to primes splitting in the extension $E/F$ whereas, in [BBJ18], we prove vertical distribution relations for these cycles at primes remaining inert in $E/F$.

### I.2.2 The Hecke side

#### I.2.2.1 The ring of $\mathbb{U}$-operators

The notion of "successor" operators introduced by Cornut–Vatsal (denoted by $T_P^u$ in the $\mathbf{GL}_2$ case [CV07, 6.3] and denoted by $\mathcal{U}_V\mathcal{U}_W$ in the case $\mathbf{U}(3) \times \mathbf{U}(2)$ [BBJ18, §3.2]) is important in our study of distribution relation.

In Chapter III, we propose a generalization of these operators and prove their integrality over the *spherical* Hecke algebra generalizing (I.3) and [BBJ18, Lemma 3.3]. Their integrality will intervene crucially in our construction of horizontal and vertical norm-compatible systems of cycles in certain general Shimura varieties. This generalization is a purely group theoretic result, that goes beyond the framework of Shimura varieties or Euler systems.

Let $F$ a local $p$-adic field, $\mathcal{O}_F$ its ring of integers, $\varpi$ a uniformizer and $k_F$ the residue field. Let $\mathbf{G}$ be a reductive group over $F$. To ease the reading of this introductory section, assume that $\mathbf{G}$ is $F$-split (this assumption will be dropped in the chapters III, IV and V). Fix a split maximal torus $\mathbf{T}$, a Borel subgroup $\mathbf{B} = \mathbf{T} \cdot \mathbf{U}^+$ with unipotent radical $\mathbf{U}^+$.

Write $\mathbf{B}^- = \mathbf{T} \cdot \mathbf{U}^-$ for the opposite Borel subgroup. Let $\mathbf{N}$ be the normalizer of $\mathbf{T}$ in $\mathbf{G}$, $W = \mathbf{N}(F)/\,\mathbf{T}(F)$ the Weyl group and let $\mathbb{A}(\mathbf{G}, \mathbf{T})$ be the standard apartment.

Being split, the group $\mathbf{G}$ has an integral model over $\mathcal{O}_F$ [BT67, §5]. Therefore, by abuse of notation, we denote also by $\mathbf{G}, \mathbf{T}$ and $\mathbf{U}^+$ the corresponding integral models $\mathcal{O}_F$. Consider the special maximal compact open subgroup $K = \mathbf{G}(\mathcal{O}_F)$. There is a reduction map $\mathrm{red}\colon \mathbf{G}(\mathcal{O}_F) \to \mathbf{G}(k_F)$. Let $I$ be the Iwahori subgroup that is defined [Tit79, 3.7] by

$$I = \{g \in \mathbf{G}(\mathcal{O}_F) : \mathrm{red}(g) \in \mathbf{B}(k_F)\}.$$

To ease notation, write $\varpi^\mu$ for $\mu(\varpi) \in \mathbf{T}(F)$ for $\mu \in X_*(\mathbf{T})$. The map $\mu \mapsto \varpi^\mu$ induces an isomorphism $\Lambda_T := X_*(\mathbf{T}) \simeq \mathbf{T}(F)/\mathbf{T}(\mathcal{O}_F)$. We have a group homomorphism $\nu\colon \mathbf{N}(F) \to \mathrm{Aff}(\mathbb{A}(\mathbf{G}, \mathbf{T}))$ (see Lemma above II.3.2.2) and $\widetilde{W}_{\mathrm{aff}} = \nu(\mathbf{N}(F))$ is called the extended affine Weyl group. It has the following decomposition

$$\widetilde{W}_{\mathrm{aff}} = \Lambda_T \rtimes W \simeq \mathbf{N}_{\mathbf{G}}(\mathbf{T})(F)/\mathbf{T}(\mathcal{O}_F),$$

where, $\nu(\mathbf{T}(F)) \simeq \mathbf{T}(F)/\mathbf{T}(\mathcal{O}_F)$, is its subgroup of translations.

Let $\mathcal{H}_I$ (resp. $\mathcal{H}_K$) denote the Iwahori–Hecke (resp. the Hecke) algebra that is the convolution algebras of locally constant compactly supported $\mathbb{Z}$-valued function on $\mathbf{G}(F)$, that are $I$-(resp. $K$-)biinvariant. The algebra $\mathcal{H}_I$ has the following $\mathbb{Z}$-basis $\{i_{\varpi^\mu w} := 1_{I\varpi^\mu w I}$ for all $\varpi^\mu w \in \Lambda_T \rtimes W\}$ (see §III.6).

In §III.8, generalizing the approach of [HKP10] to any reductive group, we define [8] and study the Bernstein and Satake untwisted homomorphisms $\dot{\Theta}_{\mathrm{Bern}}$ and $\dot{\mathcal{S}}_M^G$, respectively. A key result for the study of $\mathbb{U}$-operators is the following compatibility between the "integral part" of these two homomorphisms (see Theorem III.12.0.1 and Corollary III.12.0.1) i.e., the following diagram of $\mathbb{Z}$-modules is commutative:

$$\begin{tikzcd}
\mathcal{H}_K \arrow[dr, "\dot{\mathcal{S}}_M^G", "\simeq"'] & \\
& \mathbb{Z}[\Lambda_T]^{(W,\bullet)} \arrow[dl, "\simeq"', "\dot{\Theta}_{Bern}"] \\
\dot{\Theta}_{Bern}(\mathbb{Z}[\Lambda_T]^{(W,\bullet)}) \arrow[uu, "-*_I \mathbf{1}_K", "\simeq"'] &
\end{tikzcd}$$

where, $\mathbb{Z}[\Lambda_T]^{(W,\bullet)}$ denotes a $\mathbb{Z}$-submodule of $\mathbb{Z}[\Lambda_T]$ generated by $\{r_\mu = \sum_{w \in W/W_\mu} w \bullet \mu\colon \mu \in \mathbb{Z}[\Lambda_T]^+\}$ and $\dot{\Theta}_{Bern}(\mathbb{Z}[\Lambda_T]^{(W,\bullet)}) \subset Z(\mathcal{H}_I(\mathbb{Z}[q^{-1}]))$ (see §III.10 for more explicit details).

[8] using the right $\mathcal{H}_I(\mathbb{Z})$-module of universal unramified principal series $\mathcal{M}_I(\mathbb{Z})$.

Let $\mathbb{U} \subset \mathcal{H}_I$ be the subring generated by $i_{\varpi^\mu}$, $\mu \in \Lambda_T^+$. We make $\mathbb{U}$ acts on $\mathbb{Z}[\mathbf{G}(F)/K]$ using an embedding $\mathbb{Z}[\mathbf{G}(F)/K] \hookrightarrow \mathbb{Z}[\mathbf{G}(F)/I]$ (Lemma III.14.2.1). Among other results proven in chapter III, the following theorem (which generalizes [BBJ18, Lemma 3.3]) is the key for the construction of norm-compatible systems of cycles we intend to present:

**Theorem** I.2.2.1 (Corollaries III.14.2.1 & III.14.3.1). *There exists an embedding of rings*

$$\mathbb{U} \hookrightarrow End_{\mathbb{Z}[\mathbf{B}(F)]}\mathbb{Z}[\mathbf{G}(F)/K],$$

*such that the image of $\mathbb{U}$ is integral over the Hecke algebra $\mathcal{H}_K$.*

**Remark** I.2.2.1. *This result generalizes the property I.3 in two directions: Firstly, the linear algebraic group $\mathbf{GL}_2$ is replaced by any reductive group $\mathbf{G}$. Secondly, the hyperspecial maximal compact open subgroup $\mathbf{GL}_2(\mathbb{Z}_\ell)$ is replaced by any maximal parahoric subgroup of $\mathbf{G}(F)$. This settles completely the Hecke side, and allows one to even work over ramified places when proving distribution relations.*

### I.2.2.2 Seed Relations

In §IV we give a proof of the fact that the $\mathbf{U}$-operator attached to $\mu \in X_*(\mathbf{T})$ is a right root of the corresponding Hecke polynomial (Definition IV.3.0.1):

**Theorem** I.2.2.2. *Let $\mu \in X_*(\mathbf{T})$ be a $\mathbf{B}$-dominant cocharacter of $\mathbf{T}$. The operator $\dot{\Theta}_\mu = i_{\varpi^\mu} \in \mathbb{U}$ is a right root of the Hecke polynomial $H_{\mathbf{G},[\mu]}$ in $\mathrm{End}_P(\mathcal{C}_c(G/K,R))$.*

When $\mu$ is minuscule, this annihilation is a "lift" (in the sense of §IV.6) of a result due to Bültel [Bül97, 1.2.11]. The above theorem has the potential for offering new insights into topics beyond Euler system such as the conjectural generalization of the Eichler–Shimura congruence relation proposed by Blasius–Rogawski [BR94, §6]. Exploring this direction is the subject of a forthcoming paper.

### I.2.2.3 Geometric interpretation

Let $\mathcal{B}(G)_{\mathrm{red}}$ denote the reduced Bruhat–Tits building of $\mathbf{G}$ and, $\mathcal{A} \subset \mathcal{B}(G)_{\mathrm{red}}$ be the apartment corresponding to the split maximal torus $\mathbf{T}$. Let $\mathfrak{a} \in \mathcal{B}(G)_{\mathrm{red}}$ be the unique alcove of the reduced building fixed by the Iwahori subgroup $I$. Let $a_\circ$ be the special vertex corresponding to $K$, it belongs to the closure of $\mathfrak{a}$. Let $\mathcal{C}$ be the unique vectorial (spherical) chamber containing $\mathfrak{a}$ with apex $a_\circ$ and $\overline{\mathcal{C}}$ the opposite vectorial chamber.

In §V.1, we will use the notion of retraction to translate the purely group theoretic $\mathbb{U}$ operators into a more combinatorial fashion. This will provide a new ring $\mathcal{U}$ of geometric operators on the set of special vertices justifying why $\mathbb{U}$-operators may be thought of as a conceptual generalization of the successor operators for trees with a marked point as in the $\mathbf{GL}_2$ case (denoted by $T_P^u$ in [CV07, 6.3]) and the case $\mathbf{U}(3) \times \mathbf{U}(2)$ (denoted by $\mathcal{U}_V \mathcal{U}_W$ in [BBJ18, §3.2]).

In §V.2, we present an alternative geometric point of view for the geometric operators ring $\mathcal{U}$ using the notation of filtrations, this was suggested by C. Cornut.

#### I.2.2.4 Compatible systems of vertices

Consider now a subgroup $H$ of isometries of $\mathcal{B}(G)_{\mathrm{red}}$. In §V.3.2, we generalize the series of Lemmas proved in [BBJ18, §3.2] which gives an alternative approach for the main theorem of *loc. cit.* (see Remark V.3.2.1). The main result of §V.3.2 is:

**Theorem** I.2.2.3. *Under a technical assumption **(Mfng)** (See §V.3.2), there exists a sequence $\{a_n\}_{n\in\mathbb{N}}$ (with $a_0 = a_\circ$) of vertices in the chamber $\overline{\mathcal{C}}$, such that*

$$\sum_{k=0}^{|W|} A_k \operatorname{Tr}_{n+k,n}(a_n) = 0 \qquad (\operatorname{Tr}_{n+k,n} := \sum_{h\in H_{n+k}/H_n} h),$$

*where $A_k \in \mathcal{H}_K$ and $H_n$ is the stabilizer in $H$ of the geodesic segment $[a_1, a_n]$. We call $\{a_n\}_{n\in\mathbb{N}}$ a $H$-norm compatible system of vertices.*

### I.2.3 The Gan–Gross–Prasad setting

Gan, Gross and Prasad formulated some conjectures[9] relating special values of derivatives of automorphic $L$-functions to heights of certain special cycles on Shimura varieties constructed from embeddings of reductive groups, e.g. [GGP09, Conjecture 27.1]. In this thesis, we consider the case of special cycles on higher-dimensional Shimura varieties, where the embedding $\mathrm{Res}_{E/\mathbb{Q}} \mathbf{G}_{m,E} \hookrightarrow \mathbf{GL}_{2,\mathbb{Q}}$ defining Heegner points is replaced by an embedding of unitary groups $\mathbf{U}(n-1,1) \hookrightarrow \mathbf{U}(n,1) \times \mathbf{U}(n-1,1)$ generalizing the situation considered in §I.2.1.

[9]These conjectures may be thought of as Gross–Zagier-type formulas the cycles being generalizations of classical Heegner points.

(§VI.2) Let $E$ be a CM field, that is, an imaginary quadratic extension of a totally real number field $F$. Set $[E:\mathbb{Q}] = 2[F:\mathbb{Q}] = 2d$. Let $\tau$ be the non-trivial element of $\mathrm{Gal}(E/F)$. Fix an integer $n > 1$. Let $W$ be a Hermitian $E$-space of dimension $n$ and of signature $(n-1,1)$ at one fixed distinguished embedding $\iota\colon E \hookrightarrow \mathbb{C}$ and, of signature $(n,0)$ at the other archimedean places. Let $D$ be a positive definite Hermitian $E$-line. Consider the $n+1$-dimensional Hermitian $E$-space $V = W \oplus D$, it has signature $(n,1)$ at the distinguished archimedean place and, signature $(n,0)$ at the other ones.

We consider the $F$-algebraic reductive groups of unitary isometries $\mathbf{U}(V)$ and $\mathbf{U}(W)$. Set $\mathbf{G}_V := \mathrm{Res}_{F/\mathbb{Q}}\,\mathbf{U}(V)$ and $\mathbf{G}_W := \mathrm{Res}_{F/\mathbb{Q}}\,\mathbf{U}(W)$. We identify $\mathbf{G}_W$ with the subgroup of $\mathbf{G}_V$. Let $\mathbf{G} = \mathbf{G}_V \times \mathbf{G}_W$ and $\mathbf{H} = \Delta(\mathbf{G}_W) \subset \mathbf{G}$, where $\Delta$ denotes the diagonal embedding $\Delta\colon \mathbf{G}_W \hookrightarrow \mathbf{G}$.

(§VI.4) Let $\mathcal{X}_V$ be the Hermitian symmetric domain consisting of negative definite lines in $V \otimes_{F,\iota} \mathbb{R}$ and similarly let $\mathcal{X}_W$ be the set of negative definite lines in $W \otimes_{F,\iota} \mathbb{R}$. Setting $\mathcal{X} = \mathcal{X}_V \times \mathcal{X}_W$, the diagonal embedding $W \hookrightarrow V \oplus W$ induces an embedding of Hermitian symmetric domains $\mathcal{X}_W$ into $\mathcal{X}$; set $\mathcal{Y}$ for the image of $\mathcal{X}_W$.

(§VI.7) The two pairs $(\mathbf{G}, \mathcal{X})$ and $(\mathbf{H}, \mathcal{Y})$ are Shimura data. For small enough compact open subgroup $K_{\mathbf{G}} \subset \mathbf{G}(\mathbb{A}_f)$ (resp. $K_{\mathbf{H}} \subset \mathbf{G}(\mathbb{A}_f)$), the Shimura variety $\mathrm{Sh}_{K_{\mathbf{G}}}(\mathbf{G}, \mathcal{X})$ (resp. $\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, \mathcal{Y})$) is a complex quasi-projective smooth variety whose $\mathbb{C}$-points are given by

$$\mathbf{G}(\mathbb{Q})\backslash(\mathcal{X} \times (\mathbf{G}(\mathbb{A}_f)/K_{\mathbf{G}})) \quad (\text{resp. } \mathbf{H}(\mathbb{Q})\backslash(\mathcal{Y} \times (\mathbf{H}(\mathbb{A}_f)/K_{\mathbf{H}}))),$$

where $\mathbf{G}(\mathbb{Q})$ (resp. $\mathbf{H}(\mathbb{Q})$) acts diagonally on $\mathcal{X} \times (\mathbf{G}(\mathbb{A}_f)/K_{\mathbf{G}})$ (resp. $\mathcal{Y} \times (\mathbf{H}(\mathbb{A}_f)/K_{\mathbf{H}})$). In fact, these varieties are defined over the reflex field $E = E(\mathbf{G}, \mathcal{X}) = E(\mathbf{H}, \mathcal{Y})$ (See §VI.5 for the calculation of the reflex field).

(§VII.1.2.2) For $\star \in \{W, V\}$, we fix any compact open subgroups $K_\star \subset \mathbf{U}_\star(\mathbb{A}_{F,f})$. There exists a finite set $S$ of places of $F$ (§VII.1.2) such that $K_\star$ is of the form $K_{\star,S} \times K_\star^S$ where $K_{\star,S}$ is some compact open subgroup of $\mathbf{U}_\star(\mathbb{A}_{F,f}^S)$ and $K_\star^S$ is the product of the hyperspecial compact open subgroups $K_{\star,v} := \underline{\mathrm{U}}_\star(\mathcal{O}_{F_v}) \subset \underline{\mathrm{U}}_\star(F_v)$ for all $v \notin S$. In particular, $K_{W,v} = K_{V,v} \cap \underline{\mathrm{U}}_W(F_v)$. Set $K_v := K_{V,v} \times K_{W,v}$.

(§VI.11& §VI.12) For every $g \in \mathbf{G}(\mathbb{A}_f)$, we will denote by $\mathfrak{z}_g$ the $n$-codimensional $\mathbf{H}$-special cycle $[\mathcal{Y} \times gK] \subset \mathrm{Sh}_K(\mathbf{G}, \mathcal{X})(\mathbb{C})$, as defined in Definition VI.12.0.1. Set,

$$\mathcal{Z}_{\mathbf{G},K}(\mathbf{H}) := \{\mathfrak{z}_g : g \in \mathbf{G}(\mathbb{A}_f).\}$$

The natural map $\mathbf{G}(\mathbb{A}_f) \to \mathcal{Z}_{\mathbf{G},K}(\mathbf{H})$ given by $g \mapsto \mathfrak{z}_g$, induces the bijection

$$\mathcal{Z}_{\mathbf{G},K}(\mathbf{H}) \simeq \mathbf{H}(\mathbb{Q})Z_{\mathbf{G}}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}_f)/K,$$

where $Z_{\mathbf{G}} \simeq \mathbf{T}^1$ denotes the center of $\mathbf{G}$.

(§VI.13) The $\mathbf{H}$-special cycles $\mathcal{Z}_{\mathbf{G},K}(\mathbf{H})$ are all defined over the transfer class field $E(\infty)$ (§VI.14).

(§VI.15) The Galois group $\mathrm{Gal}(E(\infty)/E)$ acts on the set of special cycles *through* the left action of $H(\mathbb{A}_f)$. More precisely, for every $\sigma \in \mathrm{Gal}(E(\infty)/E)$, we let $h_\sigma \in \mathbf{H}(\mathbb{A}_f)$ be any element verifying $Art^1_E(\det(h_\sigma) \cdot \mathbf{T}^1(\mathbb{Q}) = \sigma|_{E(\infty)}$. For every $g \in \mathbf{G}(\mathbb{A}_f)$, we have

$$\sigma(\mathfrak{z}_g) = \mathfrak{z}_{h_\sigma g}.$$

The ultimate goal of the thesis is to construct a family of cycles verifying a tame/vertical distribution relations at inert and split places, respectively.

## I.2.4 Main theorems on distribution

Set $\mathcal{P}_{sp}$ for the set of primes of $F$ that are split in $E/F$ and do not belong to $S$ (see §VII.1.2.3). Denote by $\mathcal{N}_{sp}$ the set of square free products of primes in $\mathcal{P}_{sp}$. For every place $v$ in $\mathcal{P}_{sp}$ corresponding to the prime ideal $\mathfrak{p}_v \in \mathcal{N}_{sp}$, let $w$ be the place of $E$ defined by the embedding $\iota_v \colon \overline{F} \to \overline{F}_v$ fixed in §VI.1. We denote by $\mathfrak{P}_w$ the prime ideal of $\mathcal{O}_E$ above $\mathfrak{p}_v$ corresponding to the place $w$, and set $\mathrm{Fr}_w$ for the corresponding *geometric* Frobenius[10]. Let $\mathbf{Frob}_w \in \mathbf{T}^1(\mathbb{A}_f)$ be any element such that $\mathrm{Art}^1_E(\mathbf{Frob}_w)|_{E(\infty)^{un,w}} = \mathrm{Fr}_w$, where $E(\infty)^{un,w}$ is the maximal unramified at $w$ extension in $E(\infty)$.

### I.2.4.1 Congruence relations

As we have pointed out in the Heegner point case, the horizontal norm compatibility relation above are derived from local divisibility results (see Lemma VII.2.5.2 and Theorem VII.1.9.1) generalizing the congruence equality (I.5):

[10]Which induces $x \mapsto x^{-q_v}$ on the residue fields of $E_v^{un}$ (resp. $E_w^{un}$ and $E_{\overline{w}}^{un}$).

**Theorem** I.2.4.1. *With the above notation, we have*

$$H_w(\boldsymbol{Frob}_w)([1]_v) \equiv 0 \mod q_v^{n-1}(q_v - 1) \quad \text{in } \mathbb{Z}[q_v^{-1}][\mathbf{H}^{\mathrm{der}}(F_v)\backslash \mathbf{G}(F_v)/K_v],$$

*where $H_w$ is the Hecke polynomial attached to $\mathrm{Sh}_K(\mathbf{G}, \mathcal{X})$ at the place $w$ of the reflex field $E = E(\mathbf{G}, \mathcal{X})$ (see VII.2.3).*

### I.2.4.2 Tame relations

As a corollary of Theorem VII.1.9.1, we obtain local horizontal relations in Corollary VII.2.5.1, from which we derive the tame relations.

In §VII.1.2.3, we fix a cycle $\xi_1 := \mathfrak{z}_{g_0}$, for any $g_0$. There exists a field $\mathcal{K}$ (§VII.1.3) over which the base cycle $\mathfrak{z}_{g_0}$ is defined and such that:

**Theorem** I.2.4.2 (Horizontal relations). *There exists a collection of cycles $\xi_{\mathfrak{f}} \subset \mathbb{Z}[q_v^{-1}][\mathcal{Z}_{\mathbf{G},K}(\mathbf{H})]$ (for all $\mathfrak{f} \in \mathcal{N}_{sp}$) each defined over $\mathcal{K}(\mathfrak{f})$ (constructed in §VII.3.1) such that for every place $v \in \mathcal{P}_{sc}$, with $\mathfrak{p}_v \nmid \mathfrak{f}$, we have*

$$H_w(\mathrm{Fr}_w) \cdot \xi_{\mathfrak{f}} = \mathrm{Tr}_{\mathcal{K}(\mathfrak{p}_v\mathfrak{f})/\mathcal{K}(\mathfrak{f})}\, \xi_{\mathfrak{p}_v\mathfrak{f}},$$

*where, $H_w$ is the Hecke polynomial attached to $\mathrm{Sh}_K(\mathbf{G}, \mathcal{X})$ at the place $w$ of the reflex field $E = E(\mathbf{G}, \mathcal{X})$ defined by $\iota_v$.*

**Remark** I.2.4.1 (Vertical relations). *In a forthcoming paper we also prove vertical norm-compatible systems. Using the integrality of $\mathbb{U}$-operators one gets a result on distribution relations similar to Theorem I.2.2.3: for every place $v \in \mathcal{P}_{sp}$, there exists a family of cycles $\xi_{v,m} \in \mathcal{Z}_{\mathbf{G},K}(\mathbf{H})$, $m \in \mathbb{N}$, such that*

1. *The cycle $\xi_{v,m}$ is defined over the field $\mathcal{K}(\mathfrak{p}_v^m)$.*
2. *For $m \geq 0$, one has $\sum_{i=0}^{n(n+1)} C_i \,\mathrm{Tr}_{\mathcal{K}(\mathfrak{p}_v^{m+i})/\mathcal{K}(\mathfrak{p}_v^m)}\, \xi_{v,m+i} = 0 \quad \in \mathbb{Z}[q_v^{-1}][\mathcal{Z}_{\mathbf{G},K}(\mathbf{H})]$, where, the $C_i$s are fixed operators in the local spherical Hecke algebra.*

*We will also prove tame/vertical norm relations for inert primes of $F$. We believe that this treatment should also give similar results for other embeddings of Shimura data.*

# I.3 Applications

## I.3.1 Universal norms for Conjugate-Dual Galois Representations of $\mathrm{Gal}_{\mathcal{K}}$

Obtaining the vertical distribution relations for our norm-compatible systems in the inert case allows to prove results on the Bloch–Kato conjecture over the anticyclotomic $\mathbb{Z}_p$-extension for automorphic forms on the considered group $\mathbf{G}$. These results are analogues of a form of the Iwasawa main conjecture formulated by Perrin-Riou in [PR87] which relates Heegner points to the Selmer group of an elliptic curve over the anticyclotomic $\mathbb{Z}_p$-extension of $\mathcal{K}$.

More precisely, let $p \in \mathcal{P}_{sp}$ and let $\pi$ be the distinguished automorphic representation as in [GGP09, §27] that is unramified at $p$. Assuming that $\pi$ is cohomological, one can consider the associated conjugate self-dual $\lambda$-adic Galois representations $(\rho_{\pi,\lambda}, V)$ constructed in the middle-degree cohomology of $\mathrm{Sh}_K(\mathbf{G}, X)$ attached to $\pi$. Here, $V$ is a $L$-vector space with $\dim_L V = n(n+1)$ and $L$ is a sufficiently large finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_\lambda$. Let $T \subset V$ be a Galois stable lattice and set $W = T \otimes L/\mathcal{O}_\lambda$.

Let $\mathcal{K}_\infty$ be the anticyclotomic $\mathbb{Z}_p$-extension of $\mathcal{K}$ and let $\mathcal{K}_n$ be the $n^{\text{th}}$ intermediate finite extension in the anticyclotomic tower (i.e., the unique subfield of $\mathcal{K}_\infty$ of degree $p^n$ over $\mathcal{K}$). Define the discrete and the compact Bloch–Kato Selmer groups as $\mathbf{H}^1_f(\mathcal{K}_\infty, W) := \varinjlim_n \mathbf{H}^1_f(\mathcal{K}_n, W)$ and $\mathbf{H}^1_f(\mathcal{K}_\infty, T) := \varprojlim_n \mathbf{H}^1_f(\mathcal{K}_n, T)$, respectively, (here, the limits are with respect to restriction and corestriction, respectively, and the local conditions defining the Selmer group). For the sake of brevity and simplicity we chose to ignore most of the technicalities. For a sufficiently large ring of integers $\mathcal{O}$ of some number field $L$, one thus has a specialization

$$H_p(z; \pi_p) := \sum_{i=0}^{n(n+1)} C_i z^i \in \mathcal{O}[z].$$

where $C_i$'s are as in Remark I.2.4.1. Let $\mathcal{O}_\lambda$ be the completion of $\mathcal{O}$ at a choice of prime $\lambda$ above $p$, and enlarge $\mathcal{O}$ if needed until a root of $H_p(.; \pi_p)$, say $\beta$ lies in $\mathcal{O}_\lambda$. The distribution relations above gives with the same recipe in [BBJ18, §5] the following:

**Proposition** I.3.1.1. *Under the "ordinarity assumption" (that $v_p(\alpha) = 0$) one gets the* *family of norm-compatible cycles mentioned previously: For every $m \geq n(n+1)-1$, there is a cycle $\widetilde{\xi}_m \in \mathcal{O}_\lambda[\mathcal{Z}_K(\mathbf{G},\mathbf{H})]$ (depending on $\alpha$) such that*

$$\mathrm{Tr}_{m+1,m}(\widetilde{\xi}_{m+1}) = \widetilde{\xi}_m \in \mathcal{O}_\lambda[\mathcal{Z}_K(\mathbf{G},\mathbf{H})]. \tag{I.6}$$

Consider[11] the rational equivalence class $[\xi] \in \mathbf{CH}^n(\mathrm{Sh}_K(\mathbf{G},X))_\mathbb{Q}$ of the cycle $\xi \in \mathcal{O}_\lambda[\mathcal{Z}_K(\mathbf{G},\mathbf{H})]$. Using the diagonal cycle $\Delta_{\mathrm{Sh}_{K_V}\times \mathrm{Sh}_{K_W}}$ and the Chow–Künneth projector $\Delta_{4,\mathrm{Sh}_{K_V}\times \mathrm{Sh}_{K_W}} \in \mathbf{CH}^{2n-1}(\mathrm{Sh}_{K_V}\times \mathrm{Sh}_{K_W}\times \mathrm{Sh}_{K_V}\times \mathrm{Sh}_{K_W})_\mathbb{Q}$, we can construct a cohomologically trivial algebraic equivalence class:

$$[\xi]_0 := (\Delta_{\mathrm{Sh}_{K_V}\times \mathrm{Sh}_{K_W}} - \Delta_{4,\mathrm{Sh}_{K_V}\times \mathrm{Sh}_{K_W}})(\xi) \in \mathbf{CH}^n_0(\mathrm{Sh}_{K_V}\times \mathrm{Sh}_{K_W})_\mathbb{Q}.$$

Suppose now we are given a free rank $n(n+1)$ Galois-invariant irreducible submodule $T_\lambda$ of the $\pi_f$-equivariant component[12] of $\mathbf{H}^{2n-1}_{\text{ét}}(\overline{\mathrm{Sh}_K(\mathbf{G},X)},\mathcal{F}) \otimes \mathcal{O}_\lambda$. Define $z^0_m = \mathrm{cores}_{\mathcal{K}[\infty]/\mathcal{K}_\infty}\left(\mathrm{AJ}_p([\widetilde{\xi}_m]_0)\right) \in \mathbf{H}^1(\mathcal{K}_\infty, \mathbf{H}^{2n-1}_{\text{ét}}(\overline{\mathrm{Sh}_K(\mathbf{G},X)},\mathcal{F}))$, and write $z_m$ for the image of $z^0_m$ in $\mathbf{H}^1(\mathcal{K}_n, T_\lambda)$. It follows from (I.6) that the system $\{z_n\}$ is norm-compatible and hence it makes sense to define

$$z_\infty = \varprojlim_n z_n \in \mathbf{H}^1(\mathcal{K}_\infty, T_\lambda). \tag{I.7}$$

The span of $z_\infty$ gives a submodule $H_\infty := \Lambda \cdot z_\infty \subset \mathbf{H}^1_f(\mathcal{K}_\infty, T_\lambda)$.

Now, having our compatible norms, it should not be too hard to state analogs of Mazur's conjecture for them [Maz83][13]. However a good higher dimensional analogue of the latter conjecture should not only deal with "one possible" norm compatible family, but somehow it should look at all of them simultaneously, i.e. it should be an assertion about some submodule of a big Selmer group over a big Iwasawa-type algebra, maybe. A preliminary version of such a generalized conjecture would be formulated as follows:

**Conjecture** 1 (non-vanishing of the universal norms). *There exists an integer $n_0$ such that $z_n$ is non-torsion for all $n \geq n_0$.*

Write $\Lambda = \mathcal{O}_\lambda[[\mathrm{Gal}(\mathcal{K}_\infty/\mathcal{K})]]$. One has the following conjecture:

[11] For ease of reference, the *injectivity* of the Abel–Jacobi map among other crucial details and hypothesis are omited here.

[12] The Eulerian argument depends heavily on the component of the Kunneth decomposition of $\mathbf{H}^{2n-1}_{\text{ét}}(\mathrm{Sh}_K(\mathbf{G},X),\mathcal{F})_\mathbb{Q}$ in which the representation $T_\lambda$ is picked. A priori, the interesting ones lies in $\mathbf{H}^{n-1}_{\text{ét}} \times \mathbf{H}^n_{\text{ét}}$.

[13] The Mazur conjecture [Maz83] claims the non-triviality of the $p$-adic anticyclotomic Euler systems attached to special CM points on Shimura curves over totally real number fields. It was proved by Cornut and Vatsal. Proofs of Cornut and Cornut–Vatsal [Cor02, Vat03, CV05, CV07] rely on Galoisian properties of Heegner points and Andre-Oort proven case (They also propose an ergodic alternative using Ratner's theorem).

**Conjecture** 2 (Bloch–Kato over $\mathcal{K}_\infty$). *(i) The compact Selmer group $\mathbf{H}^1_f(\mathcal{K}_\infty, T)$ is a free $\Lambda$-module of rank 1.*

*(ii) The discrete Selmer group satisfies: $\mathrm{corank}_\Lambda \mathbf{H}^1_f(\mathcal{K}_\infty, W) = 1$.*

A key ingredient in the proof of the analogous conjectures for elliptic curves (see [Ber95, Cor02, Nek01, How04]) is the so-called Heegner module of universal norms, which is a free $\Lambda$-submodule of the compact Selmer group defined using a norm-compatible system of Heegner points defined over the finite extensions $\mathcal{K}_n$ in the anticyclotomic tower. They are analogue to $\{z_n\}$ defined above. Assuming Conjecture 1, i.e. the universal norms module $H_\infty \subset \mathbf{H}^1_f(\mathcal{K}_\infty, T)$ is free of rank 1, one can use that module to better understand the structure of the discrete module $\mathbf{H}^1_f(\mathcal{K}_\infty, W)$ and tackle:

**Conjecture** 3 (Bounding Selmer groups). *Let $X_\infty = \mathrm{Hom}_{\mathbb{Q}}(\mathbf{H}^1_f(\mathcal{K}_\infty, W), L/\mathcal{O}_\lambda)$ be the Pontrjagin dual of the discrete Selmer group. Then $X_\infty \sim \Lambda \oplus M \oplus M$, where $M$ is a torsion $\Lambda$-module satisfying $\mathrm{char}_\Lambda(M) \mid \mathrm{char}_\Lambda(\mathbf{H}^1_f(\mathcal{K}_\infty, T)/H_\infty)$.*

### I.3.2 Arithmetic applications via split Kolyvagin systems

Kolyvagin's original argument (see [Kol90, Kol91a, Kol91b, Kol91c, Gro91] as well as [How04]) uses the tame norm relations at inert auxiliary primes. In all these variants of the same fundamental argument, one uses the fact that the global cohomology $H^1(\mathcal{K}, \overline{T})$ (resp., the local cohomology groups $H^1(\mathcal{K}_\lambda, \overline{T})$) decompose into two eigenspaces $H^1(\mathcal{K}, \overline{T})^\pm$ (resp., $H^1(\mathcal{K}_\lambda, \overline{T})^\pm$) for the action of a complex conjugation on the residual representation $\overline{T}$.

Instead of applying global duality for the entire cohomology $H^1(\mathcal{K}, \overline{T})$, one does that for each of the eigenspaces $H^1(\mathcal{K}, \overline{T})^\pm$ and crucially uses the fact that if $\lambda$ is a Kolyvagin prime then $H^1_{ur}(\mathcal{K}_\lambda, \overline{T})^\pm$ (being isomorphic to $(\overline{T}/(\mathrm{Fr}_\lambda - 1)\overline{T})^\pm$) are both one-dimensional. In the case when the residual representation $\overline{T}$ of $G_\mathcal{K}$ need not extend to a representation of $G_F$, one can no longer apply the duality for the $\pm$-parts of the corresponding Selmer groups. If one attempts to use inert special primes $\lambda$, the local condition at $\lambda$ will no longer be one, but higher-dimensional and the same argument with global duality will no longer work.

Very recently, Jetchev, Nekovář and Skinner [JNS18] have managed to solve this problem by using split instead of inert auxiliary primes. In this case, if $w$ and $\overline{w}$ are the two places of $\mathcal{K}$ above a split place $v$ of $F$, one applies the duality simultaneously for $w$ and $\overline{w}$ where the local term in the duality becomes

$$H^1(\mathcal{K}_v, \overline{T})/H^1_{ur}(\mathcal{K}_v, \overline{T}) \oplus H^1(\mathcal{K}_{\overline{v}}, \overline{T})/H^1_{ur}(\mathcal{K}_{\overline{v}}, \overline{T}).$$

By using a suitable application of the Čebotarev density theorem, Jetchev, Nekovář and Skinner still manage to run the argument, this time avoiding completely the action of complex conjugation on the residual Galois representation and the corresponding Selmer groups. The Kolyvagin systems obtained in this manner are referred to as split Kolyvagin systems.

This application of the methods of Jetchev, Nekovář and Skinner to the Gan–Gross–Prasad setting above, relies in a key way on the major theorem proved in this thesis: the tame norm relations at split primes. This allows the construction of a split Kolyvagin system for cohomological Galois representations appearing in the middle-degree cohomology for Shimura varieties associated to certain product unitary isometry groups $\mathbf{U}(V) \times \mathbf{U}(W)$ that appear naturally in the context of the Gan–Gross–Prasad conjectures [GGP09].

# CHAPTER II

# PRELIMINARIES

The construction of norm-compatible families of special cycles that we intend to present here relies on a ring of $\mathbb{U}$-operators. The definition and the study of this latter ring will require very different tools from the theory of Bruhat–Tits. In this chapter, we set the necessary notations, recall some background and give a concise reminder on some of the key objects. We will also have the opportunity to provide relevant references for more complete treatments. Proofs (resp. Remarks) will be ended by $\square$ (resp. $\triangle$).

## II.1 Generalities on reductive groups: absolute case

### II.1.1 Algebraic groups

Let $k$ be a field of characteristic 0 and let $\overline{k}/k$ be a fixed choice of an algebraic closure. Let $\mathbf{G}$ be an algebraic group over $k$ [1]. We will be primarily interested in reductive/semisimple groups.

By [Mil17a, Corollary 4.10], we know that $\mathbf{G}$ admits a faithful finite-dimensional representation. Using such a representation, one can identify the semisimple and unipotent parts of elements of $\mathbf{G}(\overline{k})$[2]. The unipotent radical $R_u(\mathbf{G})$ is the largest connected normal unipotent[3] subgroup of $\mathbf{G}$. The group $\mathbf{G}$ is said to be reductive if $R_u(\mathbf{G})$ is trivial. The largest connected solvable normal subgroup of $\mathbf{G}$ is called the (solvable) radical $R(\mathbf{G})$. The group $\mathbf{G}$ is said to be semisimple[4] when $R(\mathbf{G})$ is trivial. Since unipotent groups are solvable, we have $R_u(\mathbf{G}) \subset R(\mathbf{G})$. If $\mathbf{H}$ is an algebraic subgroup of an algebraic group $\mathbf{G}$, denote by $N_{\mathbf{G}}(\mathbf{H})$ and $Z_{\mathbf{G}}(\mathbf{H})$ its normalizer and centralizer in $\mathbf{G}$, respectively. In addition, $Z_{\mathbf{G}} := Z_{\mathbf{G}}(\mathbf{G})$ will denote the center of $\mathbf{G}$.

*From now on, we will fix a reductive group $\mathbf{G}$ defined over $k$ until the end of §II.2.*

[1] That is, by definition, a reduced affine group scheme of finite type over $k$.

[2] For every $g \in \mathbf{G}(\overline{k})$, there exist unique elements $g_s, g_u \in \mathbf{G}(\overline{k})$ such that, for all representations $(V, r)$ of $\mathbf{G}$, $r(g_s)$ is semisimple and, $r(g_u)$ is unipotent. Furthermore, $g = g_s g_u = g_u g_s$ [Mil17a, §9.20].

[3] A subgroup of $\mathbf{G}$ is unipotent if $\overline{k}$-points consists of unipotent elements (have trivial semisimple part).

[4] Actually, semisimpleness (resp. reductiveness) requires the geometric groups $R(\mathbf{G}_{\overline{k}})$ (resp. $R_u(\mathbf{G}_{\overline{k}})$) to be trivial, but since our field is perfect ($\mathrm{char}(k) = 0$) this is equivalent to the definition given above [Mil17a, Propositions 19.2 & 19.10]. Let us also point out that we chose to omit the notion of smoothness, since by a theorem of Cartier, algebraic groups over characteristic zero fields are automatically smooth [Mil17a, §3.23].

### II.1.2 Tori

**Definition** II.1.2.1. *A connected $k$-algebraic group $\mathbf{T}$ is called a $k$-algebraic torus (or $k$-torus) if $\mathbf{T}_{\overline{k}} \simeq \mathbb{G}^r_{m,\overline{k}}$ for some $r \in \mathbb{N}$.*

The integer $r$ is called the rank of $\mathbf{T}$ and will be denoted $\mathrm{rk}(\mathbf{T})$. A $k$-torus $\mathbf{T}$ is called $k$-split if there exists an isomorphism $\mathbf{T} \simeq \mathbb{G}^r_{m,k}$ of $k$-groups. A $k$-torus $\mathbf{T}$ is said to be anisotropic if it contains no $k$-split subtorus.

Every torus $\mathbf{T}$, has a unique maximal $k$-split subtorus $\mathbf{S}$ and a unique maximal anisotropic subtorus $\mathbf{A}$. Define the $k$-split rank of $\mathbf{T}$ to be $\mathrm{rk}_k(\mathbf{T}) := \mathrm{rk}(\mathbf{S})$. We have an isogeny $m : \mathbf{S} \times \mathbf{A} \to \mathbf{T}$, and $\mathbf{S} \cap \mathbf{A}$ is finite [Spr98, 13.2.4].

For every torus $\mathbf{T}$ in $\mathbf{G}$, both its normalizer and centralizer are also defined over $k$ [Spr98, 13.3.1], in addition, $Z_{\mathbf{G}}(\mathbf{T})$ is connected, reductive $k$-subgroup of $\mathbf{G}$ [Spr98, 15.3.2 (i)].

**Definition** II.1.2.2. *The pair $(\mathbf{G}, \mathbf{T})$ is called a split reductive pair over $k$ if $\mathbf{T}$ is a $k$-split maximal torus (Such torus exists by [Spr98, 13.3.6]).*

### II.1.3 Digression on tori $\mathrm{Res}_{A/B}\, \mathbb{G}_{m,A}$

In this section, we present a few facts about tori of the form $\mathrm{Res}_{A/B}\, \mathbb{G}_{m,A}$ for a quadratic extension $A/B$ of characteristic 0 fields. We will encounter similar tori for various choices of extensions $A/B$, for example $\mathbb{C}/\mathbb{R}$ which will be used to define Shimura varieties. We will subsequently define and compute the associated norm-restriction map, this latter will be useful for defining the Reflex norm map and Deligne's reciprocity law for tori.

#### II.1.3.1 Weil restriction - quadratic case

Let $A/B$ be a quadratic extension of characteristic 0 fields and let $\sigma$ be the non-trivial element of $\mathrm{Gal}(A/B)$, and fix a algebraic closure $B^{ac}$ of $B$ containing $A$ . Consider the $B$-torus $\mathbb{T} = \mathrm{Res}_{A/B}\, \mathbb{G}_{m,A}$. For every $B$-algebra $R$, we have $\mathbb{T}(R) = (R \otimes_B A)^\times$. Now if $R$ is a $A$-algebra, let $\overline{R}$ denote $R$ equipped with the $\sigma$-conjugate $A$-linear structure (i.e., compose $A \to \mathrm{End}(R)$ the given $A$-structure of $R$, with the map $\sigma$). The natural map

$$A \otimes_B R \to R \oplus \overline{R}, \quad a \otimes r \mapsto (ar, a^\sigma r)$$

is an isomorphism of $A$-algebras. In addition, the natural action of $\sigma$ on $A \otimes_B R$ via $a \otimes r \mapsto a^\sigma \otimes r$, transports by the above isomorphism to an action on $R \oplus \overline{R}$ sending $r \oplus r'$ to $r' \oplus r$. Thus $\mathbb{T}(R) \simeq R^\times \times \overline{R}^\times$, which shows that after a base change to $A$, we have

$$\mathbb{T}_A \simeq \mathbb{G}_{m,A} \times \mathbb{G}_{m,A},$$

where the factors are ordered in the way that $\mathbb{T}(B) = A^\times \to \mathbb{T}(A) = A^\times \times \overline{A}^\times$ is the map $a \mapsto (a, a^\sigma)$. In particular, the group $X^*(\mathbb{T}) = X^*(\mathbb{T})_A$ is generated by two characters $\chi$ and $\overline{\chi}$ such that the induced maps on points $A^\times = \mathbb{T}(B) \subset \mathbb{T}(A) \to \mathbb{G}_{m,A}(A) = A^\times$ are $a \mapsto a$ and $a \mapsto a^\sigma$, respectively.

Define the cocharacter $\mu_\mathbb{T} \colon \mathbb{G}_{m,A} \to \mathbb{T}_A$ (resp. $\overline{\mu}_\mathbb{T}$) to be the unique one such that $\overline{\chi} \circ \mu_\mathbb{T}$ is trivial and $\chi \circ \mu_\mathbb{T} = \mathrm{Id} \in \mathrm{End}(\mathbb{G}_{m,A})$ (resp. such that $\chi \circ \overline{\mu}$ is trivial and $\overline{\chi} \circ \overline{\mu}_\mathbb{T} = \mathrm{Id} \in \mathrm{End}(\mathbb{G}_{m,A})$). On $A$-points, we have $\mu_\mathbb{T} \colon A^\times \to \mathbb{T}_A(A) \simeq A^\times \times \overline{A}^\times$, is given by $a \mapsto (a, 1)$, and $\overline{\mu}_\mathbb{T}(a) = (1, a)$.

#### II.1.3.2 Norm-one tori

The absolute Galois group of $B$ acts on $\mathbb{T}(B^{ac})$ through its projection on $\mathrm{Gal}(A/B)$[5]. More precisely, for any $A$-algebra $R$ equipped with a $\mathrm{Gal}(A/B)$-structure (e.g. $A$), there is a canonical action of $\mathrm{Gal}(A/B)$ on $\mathbb{T}(R)$ defined by

$$\sigma \colon \mathbb{T}(R) \simeq R^\times \times \overline{R}^\times \mapsto R^\times \times \overline{R}^\times, \quad (a, b) \mapsto (b^\sigma, a^\sigma).$$

Now if $R$ is only a $B$-algebra, using the canonical embedding $R \hookrightarrow S = A \otimes_B R$ (this latter has a $\mathrm{Gal}(E/F)$-structure), we may identify the $R$-points of $\mathbb{T}$ with

$$\mathbb{T}(R) = \mathbb{T}(S)^{\mathrm{Gal}(A/B)} \simeq (S \times \overline{S}^\times)^{\mathrm{Gal}(A/B)} = \left\{ (s, s^\sigma) : s \in S^\times \right\}.$$

Likewise, the absolute Galois group of $B$ also acts on the character/cocharacter group of $\mathbb{T}$ through its projection on $\mathrm{Gal}(A/B)$, we actually have $\chi^\sigma = \overline{\chi}$ and $\mu_\mathbb{T}^\sigma = \overline{\mu}_\mathbb{T}$. Indeed, let $(a, b) \in \mathbb{T}(A) \simeq A^\times \times \overline{A}^\times$, we have by definition of the action $\mathrm{Gal}(A/B)$ on characters (see II.1.4)

$$\chi^\sigma(a, b) = \sigma\chi(\sigma^{-1}(a, b)) = \sigma(\chi(b^\sigma, a^\sigma)) = \sigma(b^\sigma) = b = \overline{\chi}(a, b).$$

Recall that the $\mathbb{Z}$-module $X^*(\mathbb{T})$ is generated by $\chi$ and $\overline{\chi}$, thus it is described explicitly as follows $X^*(\mathbb{T}) = \{\chi_{n_1,n_2} := n_1\chi + n_2\chi^\sigma \text{ for } n_1, n_2 \in \mathbb{Z}\} \simeq \mathbb{Z}^2$. Now we consider the

[5] There are two actions of $\mathrm{Gal}(B^{ac}/B)$ on $\mathbb{T}(B^{ac})$: the algebraic action on $\mathbb{T}$, which factors through $\mathrm{Gal}(A/B)$, and the arithmetic action, which acts on $B^{ac}$, and does not factor through $\mathrm{Gal}(A/B)$.

following homomorphism of $\mathrm{Gal}(A/B)$-module $\mathrm{Tr}^-_{A/B}\colon X^*(\mathbb{T}) \to X^*(\mathbb{T})$, given by $\psi \mapsto \psi - \psi^\sigma$. It yields the following $\mathrm{Gal}(A/B)$-submodules

$$\begin{aligned} X^\sigma(\mathbb{T}) := \ker \mathrm{Tr}^-_{A/B} &= \left\{\chi_{n_1,n_2} \in X^*(\mathbb{T})\colon \chi^\sigma_{n_1,n_2} = \chi_{n_1,n_2}\right\} \\ &= \{\chi_{m,m} \in X^*(\mathbb{T})\colon m \in \mathbb{Z}\} \end{aligned}$$

and

$$\begin{aligned} X_0(\mathbb{T}) := \mathrm{Im}\, \mathrm{Tr}^-_{A/B} &= \left\{\chi_{n_1,n_2} \in X^*(\mathbb{T})\colon \chi_{n_1,n_2} + \chi^\sigma_{n_1,n_2} = 0\right\} \\ &= \{\chi_{m,-m} \in X^*(\mathbb{T})\colon m \in \mathbb{Z}\}\,. \end{aligned}$$

Then the exact sequence

$$0 \longrightarrow X^\sigma(\mathbb{T}) \longrightarrow X^*(\mathbb{T}) \xrightarrow{\mathrm{Tr}^-_{A/B}} X_0(\mathbb{T}) \longrightarrow 0$$

induces (see Remark II.1.4.1) an exact sequence of $B$-tori

$$1 \longrightarrow \mathbb{T}^a \longrightarrow \mathbb{T} \xrightarrow{\mathbf{N}_{A/B}} \mathbb{T}^s \longrightarrow 1,$$

where, $\mathbf{N}_{A/B}\colon \mathbb{T} \to \mathbb{T}^s \simeq \mathbb{G}_{m,B}$ denotes the norm map, mapping any element of $a \otimes r \in \mathbb{T}(R) = (A \otimes_B R)^\times$ (for any $B$-algebra $R$) to the product of its images under all Galois automorphisms $aa^\sigma \otimes r^2$, or equivalently, if we use the identification $\mathbb{T}(R) \simeq \{(s, s^\sigma) : s \in S^\times\}$ where $S = A \otimes_B R$ then[6]

$$\mathbf{N}_{A/B}(s, s^\sigma) = (ss^\sigma, ss^\sigma). \tag{II.1}$$

The torus $\mathbb{T}^s \simeq \mathbb{G}_{m,B}$ is the maximal $B$-split quotient of $\mathbb{T}$, since $X^*(\mathbb{T}^s) = X^\sigma(\mathbb{T}) \simeq X^*(\mathbb{G}_{m,F})$ is the maximal submodule on which $\mathrm{Gal}(A/B)$ acts trivially. Likewise, $\mathbb{T}^a$ is the maximal anisotropic subtorus of $\mathbb{T}$ since $X^*(\mathbb{T}^a)^{\mathrm{Gal}(A/B)} = X_0(\mathbb{T})^{\mathrm{Gal}(A/B)} = \{0\}$ (see [Spr98, 13.2.4]), and $\mathbb{T}$ is generated by its $B$-subgroups $\mathbb{T}^a$ and $\mathbb{T}^s$, with intersection $\mu_2$ the subgroup of $2^{\text{th}}$ roots of unity (see below).

Let $R$ be a $B$ algebra and $S = A \otimes_B R$, using again the fixed identification $\mathbb{T}(S) \simeq S^\times \times \overline{S}^\times$ and (II.1), we obtain a description of $R$-points for the above subtori

$$\mathbb{T}^a(R) \simeq \{(s, s^\sigma)\colon s \in S^\times, ss^\sigma = 1\} \text{ and } \mathbb{T}^s(R) \simeq \{(s, s)\colon s \in S^\times\}.$$

The subtorus $\mathbb{T}^a = \ker(\mathbf{N}_{A/B}\colon \mathrm{Res}_{A/B}\, \mathbb{G}_{m,A} \to \mathbb{G}_{m,B}) = \mathrm{Res}^{(1)}_{A/B}\, \mathbb{G}_{m,A}$ is what is called the norm one torus associated to the extension $A/B$, and we will be denoted by $\mathbf{U}(1)_{A/B}$ or simply $\mathbf{U}(1)$ when the extension is clear from the context.

[6]Here, $\sigma$ acts by swapping the components. When we compute norm maps, we use the non-canonical Galois actions for algebras, the one that just swaps component, but when we acts on rational points of the groups then we should use the canonical one defined previously which is swapping components twisted $\sigma$.

#### II.1.3.3 The norm-restriction map

Let $R$ be an $A$-algebra with a $\mathrm{Gal}(A/B)$ structure. Recall the identification

$$\mathbf{U}(1)_{A/B}(R) \simeq \left\{(a,b) \in R^\times \times \overline{R}^\times : \mathbf{N}_{A/B}(a,b) = (ab,ab) = 1\right\} = \left\{(a,a^{-1}) \colon a \in R^\times\right\}$$

The canonical Galois action of $\sigma$ on $\mathbb{T}(R)$ defined above, maps an element $(a,a^{-1}) \in \mathbf{U}(1)_{A/B}(R) \subset \mathbb{T}(R)$ to $((a^{-1})^\sigma, a^\sigma)$. Accordingly, we get a canonical action on $\mathbf{U}_{A/B}(1)$.

Let $\lambda \in X_*(\mathbf{U}_{A/B}(1))$ be a cocharacter, $\lambda$ must be invariant under the action of $\mathrm{Gal}(A/B)$, thus it is defined over $A$ and factors through $\lambda \colon \mathbb{G}_{m,A} \to \mathbf{U}_{A/B}(1)_A$. Consider the norm-restriction map,

$$\mathbf{N}_{\mathbf{U}_{A/B}(1),\lambda} \colon \mathbb{T} \xrightarrow{\mathrm{Res}_{A/B}(\lambda)} \mathrm{Res}_{A/B}\,\mathbf{U}_{A/B}(1)_A \xrightarrow{\mathbf{N}_{\mathbf{A/B}}} \mathbf{U}_{A/B}(1).$$

Let $R$ be a $B$-algebra, and set $S = A \otimes_{\mathbb{Q}} R$, we have

$$\mathrm{Res}_{A/B}\,\mathbf{U}_{A/B}(1)_A(R) = \mathbf{U}_{A/B}(1)(S) = \left\{(a,a^{-1}) \colon a \in R^\times\right\}$$

therefore, if $s \in \mathbb{T}(R) = S^\times$, then

$$\mathbf{N}_{\mathbf{U}_{A/B}(1),\lambda}(s) = \mathbf{N}_{\mathbf{A/B}}\left(\lambda(s), \lambda(s)^{-1}\right) = \left(\frac{\lambda(s)}{\lambda(s)^\sigma}, \frac{\lambda(s)^\sigma}{\lambda(s)}\right).$$

summarizing, projection to the first factor shows that

$$\mathbf{N}_{\mathbf{U}_{A/B}(1),\lambda} \colon \mathbb{T}(R) \longrightarrow \mathbf{U}_{A/B}(1)(R), \qquad s \longmapsto \tfrac{\lambda(s)}{\lambda(s)^\sigma}.$$

If, in particular, $\lambda = \chi - \chi^\sigma \in X_*(\mathbf{U}_{A/B}(1)) = X_0(\mathbb{T})$, then one gets an exact sequence of $B$-tori

$$1 \longrightarrow \mathbb{T}^s \longrightarrow \mathbb{T} \xrightarrow{s \mapsto \frac{s}{s^\sigma}} \mathbf{U}_{A/B}(1) \longrightarrow 1. \tag{II.2}$$

where, $\mathbb{T}^s = \ker \mathbf{N}_{\mathbf{U}_{A/B}(1),\chi-\chi^\sigma}$. For later use, we will denote the map $\mathbf{N}_{\mathbf{U}_{A/B}(1),\chi-\chi^\sigma}$ by $\mathbf{N}_{\mathbf{U}_{A/B}(1)}$.

### II.1.4 Characters and cocharacters

**Definition** II.1.4.1. *A character (resp. cocharacter) of a $k$-algebraic group $\mathbf{H}$ is an element of*

$$X^*(\mathbf{H})_{\overline{k}} = \mathrm{Hom}_{\overline{k}}(\mathbf{H}_{\overline{k}}, \mathbb{G}_{m,\overline{k}}), \ \textit{resp.}\ X_*(\mathbf{H})_{\overline{k}} = \mathrm{Hom}_{\overline{k}}(\mathbb{G}_{m,\overline{k}}, \mathbf{H}_{\overline{k}}).$$

*Composition defines a perfect pairing*

$$\langle\,,\rangle \colon X^*(\mathbf{H})_{\overline{k}} \times X_*(\mathbf{H})_{\overline{k}} \to \mathrm{Hom}_{\overline{k}}(\mathbb{G}_{m,\overline{k}}, \mathbb{G}_{m,\overline{k}}) \simeq \mathbb{Z}.$$

More generally, for a $k$-algebra $A$, we can also define the groups $X^*(\mathbf{H})_A := X^*(\mathbf{H}_A)$ and $X_*(\mathbf{H})_A := X_*(\mathbf{H}_A)$. For example, $X^*(\mathbf{T})_k = \mathrm{Hom}_k(\mathbf{H}, \mathbb{G}_m)$ denotes the group of $k$-rational characters of $\mathbf{T}$. Since all tori split over a finite separable extension, we have an identification

$$X^*(\mathbf{T})_{\overline{k}} \simeq X^*(\mathbb{G}_m^{\mathrm{rk}(\mathbf{T})}) \simeq \mathbb{Z}^{\mathrm{rk}(\mathbf{T})},$$

over the algebraic closure $\overline{k}$. The set $X^*(\mathbf{T})_{\overline{k}}$ has a $\mathrm{Gal}(\overline{k}/k)$-module structure defined as follows: The action of an element $\sigma \in \mathrm{Gal}(\overline{k}/k)$ on any $\chi \in X^*(\mathbf{T})_{\overline{k}}$ is defined by base change of $\chi$ through $\sigma$, this action is given on $\overline{k}$-points by

$$\sigma \circ \chi = \sigma \circ \chi \circ \sigma^{-1}.$$

Observe that $\mathrm{Gal}(\overline{k}/k)$ acts trivially on $\chi$ if and only if $\chi$ is defined over $k$. From now on, we will omit the subscript $\overline{k}$ when we write the characters and cocharacters lattices, i.e. $X^*(\mathbf{T})$ (resp. $X_*(\mathbf{T})$) instead of $X^*(\mathbf{T})_{\overline{k}}$ (resp. $X_*(\mathbf{T})_{\overline{k}}$).

**Remark** II.1.4.1. *The association $\mathbf{T} \mapsto X^*(\mathbf{T})_{\overline{k}}$ is an anti-equivalence between the categories of $k$-tori and the category of Gal($\overline{k}/k$)-modules that are free of finite rank over $\mathbb{Z}$. In fact, it yields an anti-equivalence between the category of $k$-tori split by a finite Galois extensions $k'/k$ and the category of free finitely-generated Gal($k'/k$)-modules. See [Poo17, Theorem 5.5.7] for a proof. In [GD70a, X, Proposition 1.4], one can find a generalization that classify the groups of multiplicative type over $k$.*

### II.1.5 Decomposing reductive groups

Let $\mathbf{G}^{der}$ denote the derived group of $\mathbf{G}$ [Mil17a, Definition 6.16] and $\mathbf{G}^{ad} = \mathbf{G}/Z_{\mathbf{G}}$ its adjoint quotient [Mil17a, 17.62]. These groups sit in the following diagram [Mil17a, §19d]

$$\begin{tikzcd}
Z(\mathbf{G}^{der}) \arrow[r, dashed] \arrow[d, dashed] & \mathbf{G}^{der} \arrow[d] \arrow[dr] & \\
Z_{\mathbf{G}} \arrow[r] \arrow[dr] & \mathbf{G} \arrow[r] \arrow[d] & \mathbf{G}^{ad} \\
& \mathbf{G}/\mathbf{G}^{der} &
\end{tikzcd}$$

that has the following properties

- The quotient group $\mathbf{G}/\mathbf{G}^{der}$ is a torus [Mil17a, Proposition 12.46b].
- The column and row are short exact sequences of algebraic groups.
- The diagonal maps are isogenies with common finite kernel $Z(\mathbf{G}^{der}) = Z_{\mathbf{G}} \cap \mathbf{G}^{der}$ [Mil17a, 19.21 & 19.25],
- $\mathrm{rk}(\mathbf{G}^{der}) = \mathrm{rk}(\mathbf{G}) - \dim Z_{\mathbf{G}}$ [7] [Mil17a, Proposition 19.21].

**Example** II.1.5.1. *In the case of the linear algebraic group $\mathbf{GL}_n$ for some integer $n$, the above becomes:*

$$\begin{tikzcd}
\mathbb{G}_m \cap \mathbf{SL}_n = \mu_n \arrow[r, dashed] \arrow[d, dashed] & \mathbf{SL}_n \arrow[d] \arrow[dr] & \\
\mathbb{G}_m \arrow[r] \arrow[dr, "x \mapsto x^n"'] & \mathbf{GL}_n \arrow[r] \arrow[d] & \mathbf{PGL}_n \\
 & \mathbb{G}_m &
\end{tikzcd}$$

### II.1.6 Weyl group I

Let $\mathbf{T}$ be a maximal torus of $\mathbf{G}_{\overline{k}}$. The Weyl group $W(\mathbf{G}_{\overline{k}}, \mathbf{T})$ of $\mathbf{G}_{\overline{k}}$ with respect to $\mathbf{T}$ is the finite étale group scheme $\pi_0(N_{\mathbf{G}_{\overline{k}}}(\mathbf{T}))$, hence

$$W(\mathbf{G}_{\overline{k}}, \mathbf{T}) = N_{\mathbf{G}_{\overline{k}}}(\mathbf{T})/N_{\mathbf{G}_{\overline{k}}}(\mathbf{T})^\circ = N_{\mathbf{G}_{\overline{k}}}(\mathbf{T})/Z_{\mathbf{G}_{\overline{k}}}(\mathbf{T}) = N_{\mathbf{G}_{\overline{k}}}(\mathbf{T})/\mathbf{T}.$$

By definition of the group of connected components, $N_{\mathbf{G}_{\overline{k}}}(\mathbf{T})^\circ$ is the unique normal subgroup of $N_{\mathbf{G}_{\overline{k}}}(\mathbf{T})$ such that $N_{\mathbf{G}_{\overline{k}}}(\mathbf{T})/N_{\mathbf{G}_{\overline{k}}}(\mathbf{T})^\circ$ is étale. In addition, we have a connected-étale sequence

$$N_{\mathbf{G}_{\overline{k}}}(\mathbf{T})^\circ \to N_{\mathbf{G}_{\overline{k}}}(\mathbf{T}) \to \pi_0(N_{\mathbf{G}_{\overline{k}}}(\mathbf{T})).$$

This shows the first equality. The second one follows from the rigidity of tori $N_{\mathbf{G}_{\overline{k}}}(\mathbf{T})^\circ = Z_{\mathbf{G}_{\overline{k}}}(\mathbf{T})$ and the third one is a consequence of the maximality of $T$.

The finite étale group $W(\mathbf{G}_{\overline{k}}, \mathbf{T})$ acts by definition faithfully on $\mathbf{T}$ and hence, on $X^*(\mathbf{T})$ and $X_*(\mathbf{T})$ [Mil17a, 17.41].

[7] The rank (resp. $k$-rank) of $\mathbf{G}$ is the dimension of a maximal torus in $\mathbf{G}_{\overline{k}}$ (resp. the dimension of a maximal $k$-split torus in $\mathbf{G}$). They are denoted respectively $\mathrm{rk}(\mathbf{G})$ and $\mathrm{rk}_k(\mathbf{G})$.

## II.1.7 Root datum

Let $(\mathbf{G}_{\overline{k}}, \mathbf{T})$ be a split reductive pair over $\overline{k}$, and let $\mathfrak{g}_{\overline{k}}$ be the Lie algebra of $\mathbf{G}_{\overline{k}}$. Consider the restriction of the adjoint representation $\mathrm{Ad}\colon \mathbf{G} \to \mathbf{GL}_{\mathfrak{g}_{\overline{k}}}$ to $\mathbf{T}$. The induced action of $\mathbf{T}$ on $\mathfrak{g}_{\overline{k}}$ decomposes the latter into a direct sum of spaces $\mathfrak{g}_\alpha$, for the characters $\alpha \in X^*(\mathbf{T})$, where

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g}_{\overline{k}} : \mathrm{Ad}(t) \cdot X = \alpha(t)X \text{ for all } t \in \mathbf{T}\}$$

The nonzero characters of $\mathbf{T}$ that appear in the representation $(\mathfrak{g}_{\overline{k}}, \mathrm{Ad})$ are called the roots of $(\mathbf{G}, \mathbf{T})$, and the $\mathfrak{g}_\alpha$ are called the root spaces. These roots form a finite subset of $X^*(\mathbf{T})$ denoted $\Phi(\mathbf{G}, \mathbf{T})$. We have $\mathfrak{g}_0 = \mathfrak{g}_{\overline{k}}^{\mathbf{T}} = \mathrm{Lie}(\mathbf{T}) = \mathfrak{t}$. We get a decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi(\mathbf{G}_{\overline{k}}, \mathbf{T})} \mathfrak{g}_\alpha.$$

We are ready to introduce the first important structure theorem for reductive groups over $\overline{k}$ [Mil17a, 21.11].

**Theorem** II.1.7.1. *For every $\alpha \in \Phi(\mathbf{G}, \mathbf{T})$, let $\mathbf{T}_\alpha$ be the torus $\ker(\alpha) \subset \mathbf{T}$ and set $\mathbf{G}_\alpha = Z_{\mathbf{G}}(\mathbf{T}_\alpha)$.*

1. *The pair $(\mathbf{G}_\alpha, \mathbf{T})$ is split reductive over $\overline{k}$ of semisimple rank 1.*

2. *The Lie algebra of $\mathbf{G}_\alpha$ is equal to $\mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$, and $\dim_{\overline{k}} \mathfrak{g}_\alpha = 1 = \dim_{\overline{k}} \mathfrak{g}_{-\alpha}$. Moreover $\mathbb{Q}\alpha \cap \Phi(\mathbf{G}_{\overline{k}}, \mathbf{T}) = \{\pm\alpha\}$ for $\alpha \in \Phi(\mathbf{G}_{\overline{k}}, \mathbf{T})$.*

3. *We can "exponentiate" $\mathfrak{g}_\alpha$ to obtain a canonical $\overline{k}$-subgroup $\mathbf{U}_\alpha$ of $\mathbf{G}$ (the root group) isomorphic to $\mathbb{G}_a$ and normalized by $\mathbf{T}$, on which $\mathbf{T}$ acts through the character $\alpha$, i.e.*

   $$t \cdot u(a) \cdot t^{-1} = u(\alpha(t)a), \quad \forall t \in \mathbf{T}(R), \forall a \in \mathbf{G}_\alpha(R), \forall R \text{ a } \overline{k}\text{-algebra.}$$

   *In addition, $\mathrm{Lie}(\mathbf{U}_\alpha) = \mathfrak{g}_\alpha$, and a smooth algebraic subgroup of $\mathbf{G}$ contains $\mathbf{U}_\alpha$ if and only if its Lie algebra contains $\mathfrak{g}_\alpha$.*

4. *The Weyl group $W(\mathbf{G}_\alpha, \mathbf{T})$ contains exactly one nontrivial element $s_\alpha$ and it is represented by an $n_\alpha \in N_{\mathbf{G}_\alpha}(\mathbf{T})(\overline{k})$.*

5. *There exists a unique $\alpha^\vee \in X_*(\mathbf{T})$ such that*

   $$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha, \quad \forall x \in X^*(\mathbf{T}).$$

   *In addition, we have $\langle \alpha, \alpha^\vee \rangle = 2$.*

6. $\mathbf{G}_{\overline{k}} = \langle \mathbf{T}, \mathbf{U}_\alpha \in \Phi(\mathbf{G}_{\overline{k}}, \mathbf{T}) \rangle$.

**Corollary** II.1.7.1. *The system $\mathcal{R}(\mathbf{G}_{\overline{k}}, \mathbf{T}) = (X^*(\mathbf{T}), \Phi(\mathbf{G}_{\overline{k}}, \mathbf{T}), X_*(\mathbf{T}), \Phi(\mathbf{G}_{\overline{k}}, \mathbf{T})^\vee)$ is a reduced root datum*[8]*, i.e., the following conditions holds*

1. *For each $\alpha \in \Phi(\mathbf{G}_{\overline{k}}, \mathbf{T})$, $\langle \alpha, \alpha^\vee \rangle = 2$, $s_\alpha(\Phi(\mathbf{G}_{\overline{k}}, \mathbf{T})) \subset \Phi(\mathbf{G}_{\overline{k}}, \mathbf{T})$, and the group $W(\mathcal{R}(\mathbf{G}_{\overline{k}}, \mathbf{T})) := \langle s_\alpha : \alpha \in \Phi(\mathbf{G}_{\overline{k}}, \mathbf{T}) \rangle$ generated by $\{s_\alpha\}$ for $\alpha \in \Phi(\mathbf{G}_{\overline{k}}, \mathbf{T})$ is finite.*
2. *$\mathcal{R}(\mathbf{G}_{\overline{k}}, \mathbf{T})$ is reduced, i.e. if $\alpha \in \Phi(\mathbf{G}_{\overline{k}}, \mathbf{T})$ then $2\alpha \notin \Phi(\mathbf{G}_{\overline{k}}, \mathbf{T})$.*

*Proof.* This is [Mil17a, 21.12]. □

The importance of the root datum resides in a fundamental result (Theorem II.1.11.1), it says that the ordered quadruple $\mathcal{R}(\mathbf{G}_{\overline{k}}, \mathbf{T})$ contains enough information to characterise $\mathbf{G}$ over $\overline{k}$. When there is no source of confusion, we omit the split reductive pair in its notation, i.e. $\mathcal{R}$, $W(\mathcal{R})$.

**Remark** II.1.7.1. *The root groups $\{\mathbf{U}_\alpha\}_{\alpha \in \Phi(\mathbf{G}_{\overline{k}}, \mathbf{T})}$ give an alternative description of the root system of $\mathcal{R}(\mathbf{G}_{\overline{k}}, \mathbf{T})$ without involving Lie algebras, since these groups are exactly the algebraic subgroups of $\mathbf{G}_{\overline{k}}$ that are stable by conjugation by $\mathbf{T}$ and isomorphic to $\mathbb{G}_a$.*

**Remark** II.1.7.2. *We have seen in Theorem II.1.7.1 the existence of a coroot $\alpha^\vee$ for each root $\alpha$. Let us now describe briefly how this construction goes. For each $\alpha$, we have an exact sequence*

$$\mathbf{T}_\alpha \longrightarrow \mathbf{T} \xrightarrow{\alpha} \mathbb{G}_m$$

*Let $\mathbf{G}^\alpha$ denote the derived group of $\mathbf{G}_\alpha$. Then $\mathbf{G}^\alpha$ is a split semisimple group of rank $1$, and $\mathbf{T}^\alpha := (\mathbf{G}^\alpha \cap \mathbf{T})^\circ$ is a maximal torus in $\mathbf{G}^\alpha$. In addition, this torus is sent by $\alpha$ to $\mathbb{G}_m$. There is unique cocharacter $\alpha^\vee \colon \mathbb{G}_m \to \mathbf{T}^\alpha \subset \mathbf{T}$ such that $\langle \alpha, \alpha^\vee \rangle = 2$.*

*More explicitly, let $u_\alpha \colon \mathbb{G}_a \to \mathbf{U}_\alpha \subset \mathbf{G}$ given by the Theorem II.1.7.1. Identify $\mathbb{G}_a$ with $\left\{ \begin{pmatrix} 1 & \star \\ & 1 \end{pmatrix} \right\} \subset \mathbf{SL}_2$. By Jacobson-Morozov theorem [Bou75, VIII.11.2] we know that we can extend this homomorphism to a homomorphism $\mathbf{SL}_2 \to \mathbf{G}$ such that the diagonal torus is sent into $\mathbf{T}$. Now, if we compose with the character $a \mapsto \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$ we will obtain a cocharacter $\alpha^\vee \in X_*(\mathbf{T})$ that verifies $\alpha \circ \alpha^\vee = 2$.*

[8]See [GD70b, XXI §1.1] for the definition or [Mil17a, C.37].

### II.1.8 Positive and simple roots

Most of the details of the present subsection can be found in [Bou68, VI - §1] or [Mil17a, Appendix C]. Let $V := \langle \Phi(\mathbf{G}_{\overline{k}}, \mathbf{T}) \rangle \otimes_{\mathbb{Z}} \mathbb{Q}$, where $\langle \Phi(\mathbf{G}_{\overline{k}}, \mathbf{T}) \rangle \subset X^*(\mathbf{T})(\overline{k})$ denotes the $\mathbb{Z}$-linear span of $\Phi(\mathbf{G}_{\overline{k}}, \mathbf{T})$.

**Proposition** II.1.8.1. *The pair* $(\Phi(\mathbf{G}_{\overline{k}}, \mathbf{T}), V)$ *is a root system.*

*Proof.* To be a root system, the pair $(\Phi(\mathbf{G}_{\overline{k}}, \mathbf{T}), V)$ must verify the axioms [Bou68, VI - §1 Definition 1]. These properties can be extracted from Theorem II.1.7.1 and Corollary II.1.7.1, we list them here:

1. $\Phi(\mathbf{G}_{\overline{k}}, \mathbf{T})$ is finite and does not contain 0,

2. For each $\alpha \in \Phi(\mathbf{G}_{\overline{k}}, \mathbf{T})$ there exists $\alpha^\vee \in V^\vee$ such that $\langle \alpha, \alpha^\vee \rangle = 2$ and $\langle \Phi(\mathbf{G}_{\overline{k}}, \mathbf{T}), \alpha^\vee \rangle \in \mathbb{Z}$,

3. Recall from Theorem II.1.7.1(5) that for each $\alpha \in \Phi(\mathbf{G}_{\overline{k}}, \mathbf{T})$ we have a reflection on $V$ defined by $s_\alpha \colon x \mapsto x - \langle x, \alpha^\vee \rangle \alpha$, and we have $s_\alpha(\Phi(\mathbf{G}_{\overline{k}}, \mathbf{T})) = \Phi(\mathbf{G}_{\overline{k}}, \mathbf{T})$. □

The Weyl group $W(\Phi(\mathbf{G}_{\overline{k}}, \mathbf{T}))$ of the root system $(\Phi(\mathbf{G}_{\overline{k}}, \mathbf{T}), V)$ is by definition the subgroup of automorphisms generated by the reflections $s_\alpha$:

$$W(\Phi(\mathbf{G}_{\overline{k}}, \mathbf{T})) := \langle s_\alpha : \alpha \in \Phi(\mathbf{G}_{\overline{k}}, \mathbf{T}) \rangle \subset \mathbf{GL}(V).$$

Define the root hyperplanes $H_\alpha$ for $\alpha \in \Phi(\mathbf{G}_{\overline{k}}, \mathbf{T})$ to be the set of vectors in $V^\vee$ orthogonal to $\alpha$:

$$H_\alpha = \{v \in V^\vee : \langle \alpha, v \rangle = 0\}.$$

Let $\mathcal{C}$ be a Weyl chamber, that is a connected component of $V^\vee \setminus \cup_{\alpha \in \Phi(\mathbf{G}_{\overline{k}}, \mathbf{T})} H_\alpha$. Fix any $v \in \mathcal{C}$. Let $\Phi_v^+ = \{\alpha \in \Phi(\mathbf{G}_{\overline{k}}, \mathbf{T}) : \langle \alpha, v \rangle > 0\}$. Thus $\Phi_v^+$ what is called a system of positive roots [Mil17a, C.21] for $\Phi(\mathbf{G}_{\overline{k}}, \mathbf{T})$ and is independent from the choice of $v$ in the Weyl chamber $\mathcal{C}$, this is why we will denote it $\Phi_{\mathcal{C}}^+$ or simply $\Phi^+$ when there is no risk of confusion. We will write $\Phi_{\mathcal{C}}^-$ or $\Phi^-$ for those roots with $\langle \alpha, v \rangle < 0$. The map $\mathcal{C} \mapsto \Phi_{\mathcal{C}}^+$ defines a one-to-one correspondence between the set of Weyl chambers of $(\Phi(\mathbf{G}_{\overline{k}}, \mathbf{T}), V)$ and the set of systems of positive roots.

Fix a Weyl chamber $\mathcal{C}$. A root $\alpha \in \Phi_{\mathcal{C}}^+$ is said simple if it cannot be written as a sum of two roots in $\Phi_{\mathcal{C}}^+$. The set of simple roots in $\Phi_{\mathcal{C}}^+$ is called the associated simple system $\Delta_{\mathcal{C}}$

(or simply $\Delta$ if the choice of the Weyl chamber is clear from the context). We have the following properties:

1. If $\alpha \in \Delta$ then $\#(s_\alpha(\Phi^+) \cap \Phi^+) = \#(\Phi^+) - 1$.
2. The set $\Delta$ is base for $\Phi$, that is a basis for $V$ and every root is a linear combination of elements $\Delta$ with integral coefficients that are all of the same sign.
3. The reflection group generated by the reflections $\langle s_\alpha : \alpha \in \Delta \rangle = W(\Phi(\mathbf{G}_{\overline{k}}, \mathbf{T}))$.
4. The Weyl group $W(\Phi(\mathbf{G}_{\overline{k}}, \mathbf{T}))$ acts transitively on Weyl chambers or equivalently on positive systems, mapping simple roots to simple roots.

### II.1.9 Weyl groups II

In this subsection we will see how the Weyl group $W(\mathbf{G}_{\overline{k}}, \mathbf{T})$ is related to the Weyl group of the root datum $W(\mathcal{R})$, and to the Weyl group of the root system $W(\Phi(\mathbf{G}_{\overline{k}}, \mathbf{T}))$.

**Theorem** II.1.9.1. [Mil17a, 21.37 & C.30] *The Weyl group $W(\mathbf{G}_{\overline{k}}, \mathbf{T})$ is generated by the distinguished elements $s_\alpha$, $\alpha \in \Phi(\mathbf{G}_{\overline{k}}, \mathbf{T})$. Moreover, we have a natural identification between the Weyl group of the root datum $W(\mathcal{R})$ and the Weyl group of the associated root system $(\Phi(\mathbf{G}_{\overline{k}}, \mathbf{T}), V)$.*

**Corollary** II.1.9.1. *There is a canonical isomorphism $W(\Phi(\mathbf{G}_{\overline{k}}, \mathbf{T})) = W(\mathcal{R}) \to W(\mathbf{G}_{\overline{k}}, \mathbf{T})(\overline{k})$.*

*Proof.* By definition we have a natural identification between $W(\mathcal{R})$ the Weyl group of the root datum $\mathcal{R}$ and $W(\Phi(\mathbf{G}_{\overline{k}}, \mathbf{T})$ the Weyl group of the associated root system $(\Phi(\mathbf{G}_{\overline{k}}, \mathbf{T}), V)$ [Mil17a, C.30]. For the complement of the proof see [Mil17a, 21.38] and we recommend the discussion below [GH19, 1.20] and the proof of [GH19, Proposition 1.8.1]. $\square$

We then have

$$W(\mathbf{G}_{\overline{k}}, \mathbf{T})(\overline{k}) = (N_{\mathbf{G}_{\overline{k}}}(\mathbf{T})/\mathbf{T})(\overline{k}) = N_{\mathbf{G}_{\overline{k}}}(\mathbf{T})(\overline{k})/\mathbf{T}(\overline{k}),$$

where the last equality follows by Hilbert Satz 90.

### II.1.10 Borel subgroups

A Borel subgroup $\mathbf{B}$ of $\mathbf{G}_{\overline{k}}$ is a maximal closed connected solvable subgroup. Any maximal torus is contained in a Borel subgroup [Mil17a, 21.30], and conversely any Borel contains a maximal torus. Let $\mathbf{B}$ be a Borel subgroup of $\mathbf{G}_{\overline{k}}$ containing $\mathbf{T}$, then the set of roots

$$\Phi^+_{\mathbf{B}} := \{\alpha \in \Phi(\mathbf{G}_{\overline{k}}, \mathbf{T}) : \mathfrak{g}_\alpha \in \mathfrak{b} := \mathrm{Lie}(\mathbf{B})\}$$

is a system of positive roots in $\Phi(\mathbf{G}_{\overline{k}}, \mathbf{T})$. Conversely every such system arises from a unique Borel subgroup containing $\mathbf{T}$ [Mil17a, 21.32]. Therefore the Weyl group of $W(\mathbf{G}_{\overline{k}}, \mathbf{T}))$ acts simply transitively on the set $\mathfrak{B}_{\mathbf{T}}$ of all Borel subgroups that contain $\mathbf{T}$. We have a classification of all Borel subgroups in $\mathfrak{B}_{\mathbf{T}}$: For every system of positive roots $\Phi^+$, we can construct the Borel subgroup

$$\mathbf{B}_{\Phi^+} := \langle \mathbf{T}, \mathbf{U}_\alpha(\forall \alpha \in \Phi^+)\rangle \in \mathfrak{B}_{\mathbf{T}}.$$

Conversely, every Borel subgroup $\mathbf{B}$ containing $\mathbf{T}$ is equal to $\langle \mathbf{T}, \mathbf{U}_\alpha \forall \alpha \in \Phi^+_{\mathbf{B}}\rangle \in \mathfrak{B}_{\mathbf{T}}$. The previous discussion summarizes as follows: the map $\mathbf{B} \mapsto \Phi^+_{\mathbf{B}}$ is a bijection between

$$\mathfrak{B}_{\mathbf{T}} \leftrightarrow \text{ the set of positive systems of roots in } \Phi(\mathbf{G}_{\overline{k}}, \mathbf{T}).$$

For a fixed Borel pair $(\mathbf{B}, \mathbf{T})$ we have the following decompositions:

- We have an isomorphism of split solvable algebraic groups [Mil17a, 21.34]

  $$\mathbf{B}_u \rtimes \mathbf{T} \to \mathbf{B},$$

  where $\mathbf{B}_u$ is the unipotent subgroup of $\mathbf{B}$.
- For every ordering $\{\alpha_1 \prec \cdots \prec \alpha_r\} = \Phi(\mathbf{G}_{\overline{k}}, \mathbf{T})$, the multiplication map

  $$\mathbf{U}_{\alpha_1} \times \cdots \times \mathbf{U}_{\alpha_r} \to \mathbf{B}_u,$$

  is an equivariant isomorphism of $\overline{k}$-algebraic varieties with a $\mathbf{T}$-action.

**Theorem** II.1.10.1. *All Borel pairs*[9] *in* $\mathbf{G}$ *are* $\mathbf{G}(\overline{k})$*-conjugate.*

*Proof.* This is [Mil17a, 17.13]. □

[9] A Borel pair in $\mathbf{G}$, is a pair $(\mathbf{B}, \mathbf{T})$ consisting of a Borel subgroup $\mathbf{B}$ and a maximal torus $\mathbf{T}$ contained in it.

### II.1.11 Classification in the absolute case

The root datum we attached to the split pair $(\mathbf{G}_{\overline{k}}, \mathbf{T})$ determines completely the isomorphism class of $\mathbf{G}_{\overline{k}}$, in fact we have the more precise classification

**Theorem** II.1.11.1. *The map that associates the isomorphism class of $\mathbf{G}_{\overline{k}}$ to the isomorphism class of its reduced root data $\mathcal{R}$ is a bijection:*

- *Uniqueness: let $(\mathbf{G}, \mathbf{T})$ and $(\mathbf{G}', \mathbf{T}')$ be two split connected reductive pairs over $\overline{k}$, such that $\iota\colon \mathcal{R}(\mathbf{G}, \mathbf{T}) \to \mathcal{R}(\mathbf{G}', \mathbf{T}')$ is an isomorphism*[10]. *Then $\iota$ is induced by an isomorphism $\mathbf{G} \to \mathbf{G}'$ of algebraic groups sending $\mathbf{T}$ to $\mathbf{T}'$, unique up to conjugation by $\mathbf{T}(\overline{k})$ and $\mathbf{T}'(\overline{k})$.*
- *Existence: let $\mathcal{R} = (X, \Phi, X^\vee, \Phi^\vee)$ be an abstract root datum. Then there is a split connected reductive pair $(\mathbf{G}, \mathbf{T})$ over $\overline{k}$ such that $\mathcal{R} \simeq \mathcal{R}(\mathbf{G}, \mathbf{T})$.*

*Proof.* The uniqness statement is [Mil17a, 23.25] and the existence assertion is [Mil17a, 23.55]. □

**Remark** II.1.11.1. *Fix a split reductive group $\mathbf{G}$ over $\overline{k}$. Let $\mathbf{T}$ and $\mathbf{T}'$ be two different maximal split subtori of $\mathbf{G}$. By [Mil17a, 17.105] we have that $\mathbf{T}$ and $\mathbf{T}'$ are conjugate by an element in $\mathbf{G}(\overline{k})$. Therefore the corresponding root data $\mathcal{R}(\mathbf{G}, \mathbf{T})$ and $\mathcal{R}(\mathbf{G}, \mathbf{T}')$ are actually isomorphic. Hence, by Theorem II.1.11.1, the isomorphism class of $\mathbf{G}$ is determined only by $\mathbf{G}$ and not the maximal torus. This justifies the omission, from now on, of the torus in the notation of the root datum $\mathcal{R}(\mathbf{G})$.*

### II.1.12 The connected Langlands group

For every abstract root datum $\mathcal{R} = (X, \Phi, X^\vee, \Phi^\vee)$, consider the abstract dual root datum defined by $\widehat{\mathcal{R}} := (X^\vee, \Phi^\vee, X, \Phi)$.

**Definition** II.1.12.1. *Let $\mathbf{G}$ be a reductive group defined over $k$. We define the complex dual $\widehat{\mathbf{G}}$ of $\mathbf{G}$ (sometimes called the connected Langlands dual and denoted ${}^L\mathbf{G}^0$) to be the associated connected reductive algebraic group over $\mathbb{C}$, whose root datum is dual to that of*

[10] An isomorphism of root data $(X, \Phi, X^\vee, \Phi^\vee) \to (Y, \Psi, Y^\vee, \Psi^\vee)$ is a group isomorphism $X \to Y$ sending $\Phi$ to $\Psi$ whose dual $Y^\vee \to X^\vee$ sends $\Psi^\vee$ to $\Phi^\vee$.

$\mathbf{G}$, *i.e.*

$$\mathcal{R}(\widehat{\mathbf{G}}) \simeq \widehat{\mathcal{R}(\mathbf{G})}.$$

*In the case of a torus* $\mathbf{T}$ *over* $k$ *the complex dual is simply*

$$\widehat{\mathbf{T}} = X^*(\mathbf{T}) \otimes \mathbb{C}^\times.$$

*Moreover, we have by definition the isomorphisms* $X_*(\widehat{\mathbf{T}}) \simeq X^*(\mathbf{T})$ *and* $X^*(\widehat{\mathbf{T}}) \simeq X_*(\mathbf{T})$. *We note also, that the Gal*$(\overline{k}/k)$*-action on* $X^*(\mathbf{T})$ *gives* $\widehat{\mathbf{T}}$ *a structure of Gal*$(\overline{k}/k)$*-module.*

## II.2 Structure of reductive groups: relative case

Now that we have sketched the absolute theory over $\overline{k}$, we move to the relative theory which deals with aspects over the base field $k$.

### II.2.1 $k$-tori in G

A important example of $k$-tori of $\mathbf{G}$ is the connected center $\mathbf{Z}_c$, this is the maximal central torus of $\mathbf{G}$ , i.e. the largest torus in the center $Z_{\mathbf{G}}$. We give here a few properties of $\mathbf{Z}_c$ which will be very useful in the following chapter.

**Theorem** II.2.1.1. *1. The center* $Z_{\mathbf{G}}$ *is an algebraic* $k$*-group of multiplicative type contained in all maximal tori of* $\mathbf{G}$*, and its identity connected component is*

$$Z^\circ_{\mathbf{G}} = R(\mathbf{G}) = \mathbf{Z}_c\,.$$

*2. The multiplicative homomorphism* $\mathbf{Z}_c \times \mathbf{G}^{der} \to \mathbf{G}$ *is a central isogeny (i.e. the kernel is central).*

*Proof.* See [Bor91, Propositions 11.21 & 14.2]. □

**Theorem** II.2.1.2. *All maximal* $k$*-split* $k$*-tori of* $\mathbf{G}$ *are* $\mathbf{G}(k)$*-conjugate.*

*Proof.* This can be found in [Bor91, Theorem 20.9]. □

### II.2.2 Relative root system

In this subsection, let $\mathbf{S}$ be a maximal $k$-split torus of $\mathbf{G}$ [11] and let $\mathbf{S}^{der}$ be the maximal $k$-split torus $(\mathbf{S} \cap \mathbf{G}^{der})^0_{\mathrm{red}}$ of $\mathbf{G}^{der}$ [12]. Assume $\mathbf{S}^{der}$ is not trivial or equivalently, $\mathbf{S}$ is not central in $\mathbf{G}$ (this will be the case of all situations we are interested in). Pick a maximal $k$-torus $\mathbf{T} \supset \mathbf{S}$ (the existence of such a $\mathbf{T}$ follows from [Spr98, 13.3.6 (i)]).

Let ${}_k\Phi := \Phi(\mathbf{G}, \mathbf{S}) \subset X^*(\mathbf{S})$ denote the set of non-trivial $\mathbf{S}$-weights of the adjoint action of $\mathbf{S}$ on $\mathfrak{g}$, the Lie algebra of $\mathbf{G}$. This set is called the set of relative roots of $\mathbf{G}$ with respect to $\mathbf{S}$. Therefore, the Lie algebra $\mathfrak{g}$ of $\mathbf{G}$ decomposes into the direct sum

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in {}_k\Phi} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha$ is the subspace of $\mathfrak{g}$ on which $\mathbf{S}$ acts through the nontrivial character $\alpha$.

**Remark** II.2.2.1. *[Bor91, 21.1] The relative roots ${}_k\Phi \subset X^*(\mathbf{S})$ are trivial on the maximal $k$-split subtorus $\mathbf{Z}_{c,sp}$ of the maximal central $k$-torus $\mathbf{Z}_c$. Therefore, these roots can also be seen as elements in $X^*(\mathbf{S}^{der})$ or $X^*(\mathbf{S}/\mathbf{Z}_{c,sp})$. Actually, the restriction $X^*(\mathbf{S}) \to X^*(\mathbf{S}^{der})$ induces an isomorphism ${}_k\Phi \simeq \Phi(\mathbf{S}^{der}, \mathbf{G}^{der})$.*

Let $V = X^*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $V' = X^*(\mathbf{S}^{der}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

**Lemma** II.2.2.1.
1. *The $k$-torus $\mathbf{S}^{der}$ is a maximal split $k$-torus in $\mathbf{G}^{der}$.*
2. *The natural map $\mathbf{Z}_{c,sp} \times \mathbf{S}^{der} \to \mathbf{S}$ is an isogeny.*
3. *The previous isogeny induces an orthogonal decomposition*
$$V = V' \oplus (X^*(\mathbf{Z}_{c,sp}) \otimes_{\mathbb{Z}} \mathbb{Q}).$$
4. *Since the projection of ${}_k\Phi$ on $X^*(\mathbf{Z}_{c,sp}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is trivial, one can inject ${}_k\Phi$ in $V'$ and we have $\mathbb{Q} \cdot {}_k\Phi = V'$.*

*Proof.* See [Con16, 11.3.3] □

We can deduce a similar statement to Proposition II.1.8.1:

**Theorem** II.2.2.1. *The roots $\Phi(\mathbf{G}^{der}, \mathbf{S}^{der})$ generate $X^*(\mathbf{S}^{der}) \otimes_{\mathbb{Z}} \mathbb{Q}$, i.e. $V' = X^*(\mathbf{S}^{der}) \otimes_{\mathbb{Z}} \mathbb{Q}$. The two pairs $({}_k\Phi, V')$ and $\big(\Phi(\mathbf{G}^{der}, \mathbf{S}^{der}), X^*(\mathbf{S}^{der}) \otimes_{\mathbb{Z}} \mathbb{Q}\big)$ are isomorphic root systems.*

[11] This is unique up to $\mathbf{G}(k)$-conjugacy by Thereom II.2.1.1

[12] The group $\mathbf{S} \cap \mathbf{G}^{der}$ can be disconnected or non-reduced, see [Con16, Example 11.3.4].

*Proof.* The fact that $\big(\Phi(\mathbf{G}^{der}, \mathbf{S}^{der}), X^*(\mathbf{S}^{der}) \otimes_{\mathbb{Z}} \mathbb{Q}\big)$ is a root datum is [Bor91, Thm 21.6]. By Remark II.2.2.1, the same statement holds for $({}_k\Phi, V')$, so we obtain an isomorphism of root systems between the

$$({}_k\Phi, V') \simeq \big(\Phi(\mathbf{G}^{der}, \mathbf{S}^{der}), X^*(\mathbf{S}^{der}) \otimes_{\mathbb{Z}} \mathbb{Q}\big) \quad \square.$$

As opposed to the absolute case, $({}_k\Phi, V')$ can be non-reduced if $\mathbf{S}$ is not a maximal $k$-torus, and the weight spaces $\mathfrak{g}_\alpha$, for $\alpha \in {}_k\Phi$, are no longer always 1-dimensional. We define the reduced root system

$${}_k\Phi_{red} = \{\alpha \in {}_k\Phi \colon \frac{1}{2}\alpha \notin {}_k\Phi\}.$$

**Remark** II.2.2.2. *Observe that*

$$\begin{aligned} {}_k\Phi = \emptyset &\Leftrightarrow \mathbf{G} = Z_{\mathbf{G}}(\mathbf{S}) \\ &\Leftrightarrow \mathbf{S} \subset \mathbf{G} \textit{ central (e.g., } \mathbf{S} = 1 \textit{ when } \mathbf{G} \textit{ is semisimple)} \\ &\Leftrightarrow \mathbf{G} \textit{ has no proper parabolic } k\textit{-subgroups (see §II.2.6 for definition).} \end{aligned}$$

*We will then assume from now on ${}_k\Phi \neq \emptyset$.*

The Weyl group $W({}_k\Phi)$ of the root system $({}_k\Phi, V')$ is by definition the subgroup of automorphisms of $V$ generated by the reflections $s_\alpha \colon x \mapsto x - \langle x, \alpha^\vee \rangle \alpha$ :

$$W({}_k\Phi) = \langle s_\alpha : \alpha \in {}_k\Phi \rangle \subset \mathbf{GL}(V') \times \mathbf{GL}(X^*(\mathbf{Z}_{c,sp}) \otimes_{\mathbb{Z}} \mathbb{Q}) \subset \mathbf{GL}(V).$$

**Remark** II.2.2.3 (Positive and simple "relative" roots). *All of the notions and results we encountered in the "absolute" situation of §II.1.8.1 remain valid (modulo few adaptions) for the root system $({}_k\Phi, V)$. Similarly, define the set of hyperplanes $(H_\alpha)_{\alpha \in {}_k\Phi}$, Weyl chambers $WC({}_k\Phi) := \pi_0(V^\vee \setminus \cup_{\alpha \in {}_k\Phi} H_\alpha)$, systems of positive roots $\{\Phi^+_{\mathcal{C}}\}_{\mathcal{C} \in WC({}_k\Phi)}$ [Bou68, VI - §1]. For every chamber $\mathcal{C}$, let $\Delta_{\mathcal{C}}$ be the associated simple root system.*

### II.2.3 Relative root datum

By Theorem II.2.2.1 for each $\alpha \in {}_k\Phi$ there exists a cocharacter $\alpha^\vee \in X_*(\mathbf{S}^{\mathrm{der}})$ verifying some properties (listed in the proof of Proposition II.1.8.1). We identify each $\alpha^\vee$ with an element of $X_*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{Q}$ under the decomposition

$$X_*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{Q} = (X_*(\mathbf{S}^{\mathrm{der}}) \otimes_{\mathbb{Z}} \mathbb{Q}) \oplus (X_*(\mathbf{Z}_{c,sp}) \otimes_{\mathbb{Z}} \mathbb{Q}).$$

Therefore, we get an injection ${}_k\Phi \hookrightarrow X_*(\mathbf{S})$ onto a subset of non-trivial elements which we will denote ${}_k\Phi^\vee$.

**Theorem** II.2.3.1. *The quadruple* ${}_k\mathcal{R} = \mathcal{R}(\mathbf{G}, \mathbf{S}) = (X^*(\mathbf{S}), {}_k\Phi, X_*(\mathbf{S}), {}_k\Phi^\vee)$ *is a root datum.*

*Proof.* This is [Spr98, 15.3.8] □

We denote by $W({}_k\mathcal{R})$ the Weyl group of the root datum ${}_k\mathcal{R}$. It is naturally identified with the Weyl group $W({}_k\Phi)$ of the associated root system ${}_k\Phi = \Phi(\mathbf{G}, \mathbf{S})$.

### II.2.4 Relative Weyl group

Similarly to the absolute case, the quotient $N_{\mathbf{G}}(\mathbf{S})/Z_{\mathbf{G}}(\mathbf{S})$ is a finite étale $k$-group scheme. Its group of $k$-points $(N_{\mathbf{G}}(\mathbf{S})/Z_{\mathbf{G}}(\mathbf{S}))(k) = N_{\mathbf{G}}(\mathbf{S})(k)/Z_{\mathbf{G}}(\mathbf{S})(k)$[13] is called the relative Weyl group $W(\mathbf{G}, \mathbf{S})$ and is canonically isomorphic to the Weyl group $W({}_k\Phi)$ of the root system $(V, {}_k\Phi)$ [Bor91, Thm 21.2]:

$$W({}_k\mathcal{R}) = W({}_k\Phi) \xrightarrow{\sim} W(\mathbf{G}, \mathbf{S})(k).$$

### II.2.5 The relative root group $\mathbf{U}_\alpha$

We define in this subsection the root group $\mathbf{U}_\alpha \subset \mathbf{G}$ associated to each $\alpha \in {}_k\Phi$. Unlike the split case obtained over $\overline{k}$, this group may have rather large dimension, and can even be non-commutative. The existence of these groups follows from a more general construction that does not require $\mathbf{G}$ to be reductive:

**Theorem** II.2.5.1. *Let* $\mathbf{H}$ *be a smooth connected affine* $k$*-group endowed with an action of a* $k$*-split torus* $\mathbf{S}$. *Let* $\Lambda \subseteq X^*(\mathbf{S}) \setminus \{0\}$ *be a semigroup. There exists a unique, unipotent*[14], $\mathbf{S}$*-stable smooth connected* $k$*-subgroup* $\mathbf{U}_\Lambda(\mathbf{H})$ *whose Lie algebra*

$$\mathfrak{u}_\Lambda = \oplus_{\alpha \in \Lambda \cap \Phi(\mathbf{H}, \mathbf{S})} \mathfrak{g}_\alpha.$$

*Moreover, every* $S$*-stable smooth connected* $k$*-subgroup such that all* $S$*-weights on its Lie algebra occurs in* $\Lambda$ *will be contained in* $\mathbf{U}_\Lambda$.

*Proof.* This is [CGP15, Proposition 3.3.6]. □

[13] This equality follows using Hilbert Satz 90.

[14] This is ensured by the requirement $0 \notin \Lambda$.

For any $\alpha \in {}_k\Phi$ consider the following semigroup $\mathbb{Z}_{>0}\alpha \subset X^*(\mathbf{S})$. By the previous Theorem II.2.5.1 let us associate to each $\alpha \in {}_k\Phi$ the group $\mathbf{U}_\alpha := \mathbf{U}_{\mathbb{Z}_{>0}\alpha}(\mathbf{G})$, It is a smooth connected unipotent $k$-subgroup of $\mathbf{G}$ that is normalized by $\mathbf{S}$ and for which the Lie algebra $\mathfrak{u}_{\mathbb{Z}_{>0}\alpha}$ is the span of the weight spaces $\mathfrak{g}_\beta$ for $\beta \in \mathbb{Z}_{>0}\alpha \cap {}_k\Phi$. In addition, if $2\alpha \in {}_k\Phi$ then we have $\mathbf{U}_{2\alpha} \subseteq \mathbf{U}_\alpha$, this follows immediately from the theorem above which ensures that $\mathbf{U}_\alpha$ contains every smooth connected $k$-subgroup normalized by $\mathbf{S}$ and for which the $\mathbf{S}$-weights on its Lie algebra are positive integral multiples of $\alpha$. The $k$-group $\mathbf{U}_\alpha$ is called the relative root group of $\mathbf{G}$ associated to $\alpha \in {}_k\Phi$. If there is no risk of confusion with the root groups defined over $\overline{k}$ we will drop the adjective relative in this definition.

We present in the following proposition a few other properties of root groups (see [Bor91, §14.4]):

**Proposition** II.2.5.1. *a. The group $Z_{\mathbf{G}}(\mathbf{S})$ normalizes $\mathbf{U}_\alpha$ for all $\alpha \in {}_k\Phi$,*

*b. $Lie_F(\mathbf{U}_\alpha) = \mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha}$,*

*c. Let $\Psi$ be a positively closed*[15] *subset of ${}_k\Phi$. There exists a unique closed, connected, unipotent $k$-subgroup $\mathbf{U}_\Psi \subset \mathbf{G}$, that is normalized by $Z_{\mathbf{G}}(\mathbf{S})$. The product morphism*

$$\prod_{\alpha \in \Psi_{red}} \mathbf{U}_\alpha \to \mathbf{U}_\Psi,$$

*is an isomorphism of $k$-varieties, where the product is taken in any order for $\Psi$. In particular, $Lie_k(\mathbf{U}_\Psi) = \sum_{\alpha \in \Psi} \mathfrak{g}_\alpha$.*

Whenever a system of positive roots ${}_k\Phi^+$ is fixed in ${}_k\Phi$ with base $\Delta \subset {}_k\Phi_{red}$, we denote the groups $\mathbf{U}_{{}_k\Phi^+}$ (resp. $\mathbf{U}_{{}_k\Phi^-}$) simply by $\mathbf{U}^+$ (resp. $\mathbf{U}^-$).

### II.2.6 Parabolic subgroups

An algebraic subgroup $\mathbf{P}$ of $\mathbf{G}$ is said to be parabolic if the quotient $\mathbf{G}/\mathbf{P}$ is a complete algebraic variety[16]. Equivalently, a subgroup $\mathbf{P}$ is parabolic if and only if it contains a

[15]Being positively closed [Bor91, 14.7] (equivalently being special [Bor91, 14.5]) is defined as follows: an arbitrary subset $\Psi \subset {}_k\Phi$ is called positively closed, if $\Psi$ lies in an open half-space of $X^*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{Q}$ (or equivalently, belongs to a positive set of roots for some ordering of $\Phi$) and if for all $\alpha, \beta \in \Psi$ we have

$$[\alpha, \beta] := \{n\alpha + m\beta\colon \text{ for all } n, m \in \mathbb{Z}_{>0}\} \cap {}_k\Phi \subset \Psi.$$

[16]This definition is equivalent to requiring the quotient $\mathbf{G}/\mathbf{P}$ to be projective. Indeed, since we quotient by a closed subgroup, the latter quotient is a quasi-projective variety, but complete and quasi-projective implies projective, which shows the equivalence of the two definitions.

Borel subgroup (a maximal connected solvable subgroup) [Bor91, §11.2]. A Levi factor of $\mathbf{P}$ is a reductive subgroup $\mathbf{M}$ verifying

$$\mathbf{P} = R_u(\mathbf{P}) \rtimes \mathbf{M}\,.$$

**Proposition** II.2.6.1. *Let $\mathbf{S}$ be any $k$-split subtorus of $\mathbf{G}$. Its centralizer $Z_{\mathbf{G}}(\mathbf{S})$ is a Levi subgroup of a parabolic $k$-subgroup of $\mathbf{G}$.*

*Proof.* A proof can be found in [Bor91, Proposition 20.4]. □

Let ${}_k\Phi^+$ be a system of positive roots in ${}_k\Phi$ with base $\Delta \subset {}_k\Phi_{red}$ (see Remark II.2.2.3). Any subset $J \subset \Delta$ is a base for a root system ${}_k\Phi_J := \mathbb{Z}J \cap {}_k\Phi$. The subgroup $\langle Z_{\mathbf{G}}(\mathbf{S}), \mathbf{U}_{{}_k\Phi_J \cup {}_k\Phi^+} \rangle \subset \mathbf{G}$ is a parabolic subgroup, we denote it by $\mathbf{P}_J$, such a Parabolic will be called semi-standard. We have $R_u(\mathbf{P}_J) = \mathbf{U}_{{}_k\Phi^+ \setminus {}_k\Phi_J^+}$, where ${}_k\Phi_J^+ := \mathbb{Z}J \cap {}_k\Phi^+$. The Levi factor $\mathbf{M}_J$ of $\mathbf{P}_J$ is the subgroup generated by $\langle Z_{\mathbf{G}}(\mathbf{S}), \mathbf{U}_{{}_k\Phi_J} \rangle$. Actually, this Levi subgroup is the centralizer of the split torus $\mathbf{S}_J := \cap_{\alpha \in {}_k\Phi_J} \ker \alpha$. When $J = \emptyset$ we get a minimal parabolic subgroup $\mathbf{P}_\emptyset$ with Levi factor $Z_{\mathbf{G}}(\mathbf{S})$, confirming the above proposition. If $J = \Delta$, then $\mathbf{P}_\Delta = \mathbf{M}_\Delta = \mathbf{G}$ and $\mathbf{S}_\Delta$ is the unique maximal split torus of the center $Z_{\mathbf{G}}$. We refer the reader to [Bor91, §21] for proofs of the results above.

**Proposition** II.2.6.2. *All minimal parabolic $k$-subgroups are $\mathbf{G}(k)$-conjugate.*

*Proof.* This can be found in [Bor91, Theorem 20.9]. □

**Remark** II.2.6.1. *Using notations of Remark II.2.2.3, we have the following identifications:*

$$\begin{gathered} \{\textit{Minimal parabolic subgroups containing } \mathbf{S}\} \\ \updownarrow \\ \{\textit{Systems of positive roots}\} \\ \updownarrow \\ \{\textit{Weyl chambers in } ({}_k\Phi, V)\}. \quad \square \end{gathered}$$

## II.2.7 Tits system

We first recall the definition of a Tits system, which we will be regularly using in chapter III. For more details we refer to [Bou68, §2 in Chapter IV].

**Definition** II.2.7.1. *A* Tits system *(or BN-pair) is a quadruple* $(G, B, N, \mathcal{S})$ *composed of an abstract group* $G$*, two subgroups* $B$ *and* $N$*, and a subset* $\mathcal{S} \subseteq W := N/(B \cap N)$ *such that the following four axioms are fulfilled:*

- $B \cap N$ *is normal in* $N$ *and* $B \cup N$ *generates* $G$*,*
- *The elements of* $\mathcal{S}$ *have order 2 and generate* $W$*,*
- *For all* $s \in \mathcal{S}$ *and* $w \in W$*,* $sBw \subseteq BwB \cup BswB$ *(using any representatives for* $s$ *and* $w$ *in* $N$*),*
- $sBs \not\subseteq B$ *for all* $s \in \mathcal{S}$ *(this is equivalent to* $sBs \neq B$*, since* $s^2$ *is of order* $2$*).*

**Remark** II.2.7.1. *The nomenclature BN-pair for* $G$ *is justified by [Bou68, IV-§2.5 Remark 1] which asserts that the set* $\mathcal{S}$ *is uniquely determined by the triplet* $(G, B, N)$*.*

A good part of the theory developed by Borel and Tits in [BT65, BT72] may be condensed in the following fundamental theorem. Define for each $\alpha \in \Phi$ the set $M_\alpha$ of elements of $N_{\mathbf{G}}(\mathbf{S})(k)$ whose image in the Weyl group ${}_kW = N_{\mathbf{G}}(\mathbf{S})(k)/Z_{\mathbf{G}}(\mathbf{S})(k)$ is the reflection $s_\alpha$.

**Theorem** II.2.7.1. *The datum* $\left(Z_{\mathbf{G}}(\mathbf{S})(k), (\mathbf{U}_\alpha(k), M_\alpha)_{\alpha \in \Phi(\mathbf{G},\mathbf{S})}\right)$ *is a generating root datum (donnée radicielle génératrice) of type* ${}_k\Phi$ *in* $\mathbf{G}(k)$ *in the sense of Bruhat and Tits [BT72, 6.1.1.].*

The nomenclature "generating root datum" means that the above datum verifies the following list of axioms:

(DR1) The obvious fact that $Z_{\mathbf{G}}(\mathbf{S})(k)$ is a subgroup of $\mathbf{G}(k)$ and each $\mathbf{U}_\alpha(k)$ is a nontrivial subgroup of $\mathbf{G}(k)$.

(DR1') $\mathbf{G}(k)$ is generated by $Z_{\mathbf{G}}(\mathbf{S})(k)$ and the $\mathbf{U}_\alpha(k)$'s for all $\alpha \in {}_k\Phi$.

(DR2) For every $\alpha, \beta \in {}_k\Phi$ the group of commutators $[\mathbf{U}_\alpha(k), \mathbf{U}_\beta(k)]$ is generated by $\mathbf{U}_{p\alpha+q\beta}(k)$ for $p, q \in \mathbb{Z}_{>0}$ and $p\alpha + q\beta \in {}_k\Phi$.

(DR3) If $\alpha$ and $2\alpha$ are both in ${}_k\Phi$, we then have $\mathbf{U}_{2\alpha}(k) \subsetneq \mathbf{U}_\alpha(k)$.

(DR4) For $\alpha \in {}_k\Phi$, $M_\alpha$ is a right $Z_{\mathbf{G}}(\mathbf{S})(k)$-coset in $\mathbf{G}(k)$ and $\mathbf{U}_{-\alpha}(k) \backslash \{1\} \subset \mathbf{U}_\alpha(k) M_\alpha \mathbf{U}_\alpha(k)$.

(DR5) For $\alpha, \beta \in {}_k\Phi$ and $n \in M_\alpha$, we have

$$n\mathbf{U}_\beta(k)n^{-1} = \mathbf{U}_{s_\alpha(\beta)}(k).$$

(DR6) $Z_{\mathbf{G}}(\mathbf{S})(k)\mathbf{U}^+(k) \cap \mathbf{U}^-(k) = \{1\}$.

**Theorem** II.2.7.2. *Let* $\mathbf{P}$ *be a minimal parabolic* $k$*-subgroup of* $\mathbf{G}$ *containing* $\mathbf{S}$*. Let* ${}_k\Delta$ *be the base of the positive system of roots* ${}_k\Phi^+ = \Phi(\mathbf{P}, \mathbf{S})$*, and let* $\mathcal{S} := \{s_\alpha \,|\, \alpha \in {}_k\Delta\}$ *be* *the associated set of simple positive reflections. The* 4*-tuple*

$$(\mathbf{G}(k), \mathbf{P}(k), N_{\mathbf{G}}(\mathbf{S})(k), \mathcal{S}),$$

*is a saturated Tits system with Weyl group* ${}_kW$*.*

*Proof.* For the details see [BT72, Proposition 6.1.12] where $\mathbf{P}(k)$ is equal to $Z_{\mathbf{G}}(\mathbf{S})(k) \cdot \prod_{\alpha \in {}_k\Phi^+} \mathbf{U}_\alpha(k)$ by [Bor91, 14.18 & 21.11]. □

## II.3 Bruhat–Tits buildings theory

Bruhat and Tits made a profound exploration of reductive groups over local fields by constructing for them a combinatorial avatar: let $\mathbf{G}$ be a reductive group over a $F$; a finite extension of $\mathbb{Q}_p$ for some prime $p$. In their seminal work [BT72, BT84] they associate to $G^{ad} = \mathbf{G}^{ad}(F)$ an affine building $\mathcal{B}(\mathbf{G}, F)_{\mathrm{red}}$, and to $\mathbf{G}$ a building $\mathcal{B}(\mathbf{G}, F)_{\mathrm{ext}}$. The first one is called the reduced Bruhat–Tits building of $\mathbf{G}$, the latter one is called the extended Bruhat–Tits building of $\mathbf{G}$. Here is a gentle mise-en-bouche: the building $\mathcal{B}(\mathbf{G}, F)_{\mathrm{red}}$ is a complete metric space, that has a structure of a poly-simplicial complex. The building $\mathcal{B}(\mathbf{G}, F)_{\mathrm{red}}$ is obtained by "gluing" a family of distinguished subsets, called apartments. These apartments are affine spaces for some fixed real vector space. This inner poly-simplicial structure comes with an action of the group $G$, this latter acts isometrically by polysimplcial automorphisms on $\mathcal{B}(\mathbf{G}, F)_{\mathrm{red}}$. In the present subsection we will be following [Lan96],[Yu09], [SS97, §1.1] and [Vig16, §3] to introduce the main ingredients needed in the construction of the above buildings: the groups $U_{\alpha,r}$. The previous references played the role of guide to the non-initiated (me) to explore the encyclopaedic and monumental treatise [BT72, BT84] and helped extract a reasonably brief exposition.

### II.3.1 Notations

- $F$ a finite extension of $\mathbb{Q}_p$ for some prime $p$, $\mathcal{O}_F$ its ring of integers, $\varpi$ a fixed uniformizer in $\mathcal{O}_F$,
- $\omega\colon F^\times \to \mathbb{Z}$ the normalized discrete valuation, i.e. $\omega(\varpi) = 1$, and $|\cdot|_F = q^{\omega(\cdot)}$ for the corresponding normalized absolute value,
- $k_F$ the residue field of $F$, and $q$ its cardinality,
- $F^{sep}$ a fixed separable closure,
- For any connected reductive $F$-group $\mathbf{H}$, there exists a homomorphism of groups $\nu_H\colon \mathbf{H}(F) \to \mathrm{Hom}_{\mathbb{Z}}(X^*(\mathbf{H})_F, \mathbb{Z})$ characterized by
$$\langle \nu_H(h), \chi\rangle = \nu_H(h)(\chi) = -\omega(\chi(h)), \text{ for all } h \in \mathbf{H}(F) \text{ and } \chi \in X^*(\mathbf{H})_F.$$
- Fix $\mathbf{G}$ a connected reductive group over $F$, we use the convention that all reductive groups are connected,
- Let $\mathbf{Z}_c$ denote the maximal central F-torus, and $\mathbf{Z}_{c,sp}$ its maximal $F$-split $F$-subtorus,

- Let $\mathbf{S}$ be a maximal $F$-split subtorus of $\mathbf{G}$,
- For any algebraic subgroup $\mathbf{H} \subset \mathbf{G}$, we will denote by $N_{\mathbf{G}}(\mathbf{H})$ and $Z_{\mathbf{G}}(\mathbf{H})$ the normalizer and the centralizer of $\mathbf{H}$ in $\mathbf{G}$ respectively,
- The root system of $\mathbf{G}$ with respect to $\mathbf{S}$ will be denoted $\Phi := \Phi(\mathbf{G}, \mathbf{S})$, $\Phi^+$ a system of positive roots in $\Phi$, $\Delta$ the associated base of simple roots and the Weyl group $W = W(\mathbf{G}, \mathbf{S})(F) = N_{\mathbf{G}}(\mathbf{S})(F)/Z_{\mathbf{G}}(\mathbf{S})(F)$ (see §II.2.4),
- Set $\mathbf{M} := \mathbf{M}_{\emptyset}$ [17] $= Z_{\mathbf{G}}(\mathbf{S})$ for the centralizer (a minimal Levi $F$-subgroup of $\mathbf{G}$), $\mathbf{N} := N_{\mathbf{G}}(\mathbf{S})$ for the normalizer and $\mathbf{P} := \mathbf{P}_{\emptyset}$ for the minimal parabolic with Levi factor $\mathbf{M}$ and unipotent radical $\mathbf{U}^+ = \mathbf{U}_{\Phi^+}$, we have $\mathbf{P} = \mathbf{U}_{\Phi^+} \rtimes \mathbf{M}$.
- We will sometimes use the notation $\Box \otimes k = \Box_k$, for a field $k$.

## II.3.2 The standard apartment

**Lemma** II.3.2.1. *We have the following commutative diagram*

$$\begin{array}{ccccc} X^*(\mathbf{M})_F & \times & X_*(\mathbf{M})_F & \longrightarrow & \mathbb{Z} \\ \big\downarrow & & \big\| & & \big\| \\ X^*(\mathbf{S}) & \times & X_*(\mathbf{S}) & \longrightarrow & \mathbb{Z} \end{array}$$

*where only the bottom line is a perfect pairing. In addition, the first vertical embedding identifies canonically $X^*(\mathbf{M})_F$ with a finite index subgroup of $X^*(\mathbf{S})$.*

*Proof.* Any $F$-rational cocharacter in $X_*(\mathbf{M})_F$ factors through $\mathbf{S}$, since $\mathbf{S}$ is the unique[18] maximal $F$-split torus in $\mathbf{M}$, therefore $X_*(\mathbf{M})_F \simeq X_*(\mathbf{S})_F$ hence by [Spr98, Proposition 13.2.2 (i)] we get

$$X_*(\mathbf{M})_F \simeq X_*(\mathbf{S}).$$

The product map $\pi \colon \mathbf{M}^{\mathrm{der}} \times Z_{\mathbf{M}}^{\circ} \to \mathbf{M}$ given on rational points by $(x, y) \mapsto xy^{-1}$ induces an isomorphism of algebraic groups

$$(\mathbf{M}^{\mathrm{der}} \times Z_{\mathbf{M}}^{\circ})/\ker \pi \to \mathbf{M}\,.$$

We may then identify the $X^*(\mathbf{M})$ with the characters of $\mathbf{M}^{\mathrm{der}} \times Z_{\mathbf{M}}^{\circ}$ that are trivial on $\ker \pi = \{(x, x) : x \in (\mathbf{M}^{\mathrm{der}} \cap Z_{\mathbf{M}}^{\circ})\}$. Now because $\ker \pi$ is finite, the group of characters $X^*(\mathbf{M})$ identifies with a finite index subgroup of $X^*(\mathbf{M}^{\mathrm{der}} \times Z_{\mathbf{M}}^{\circ}) \simeq X^*(\mathbf{M}^{\mathrm{der}}) \oplus X^*(Z_{\mathbf{M}}^{\circ}) \simeq$

[17] see II.2.6 for notation.

[18] Since all maximal $F$-split tori of $\mathbf{M}$ are $\mathbf{M}(F)$-conjugate (Theorem II.2.1.2), and $\mathbf{M}$ is the centralizer of $\mathbf{S}$ in $\mathbf{G}$.

$X^*(Z_{\mathbf{M}}^\circ)$. The composition of the following maps

$$X^*(\mathbf{M}) \xrightarrow{\simeq} X^*\left((\mathbf{M}^{\mathrm{der}} \times Z_{\mathbf{M}}^\circ)/\ker\pi\right) \hookrightarrow X^*(\mathbf{M}^{\mathrm{der}} \times Z_{\mathbf{M}}^\circ) \xrightarrow{\simeq} X^*(Z_{\mathbf{M}}^\circ),$$

is the restriction map.

Recall that $\mathbf{M}$ is reductive (see Proposition II.2.6.1 above), so $Z_{\mathbf{M}}^\circ$ is an $F$-torus and it contains $\mathbf{S}$. Let $Z_{\mathbf{M}}^{an}$ be the maximal $F$-anisotropic subtorus of $Z_{\mathbf{M}}^\circ$, we then have an isogeny $\mathbf{S} \times Z_{\mathbf{M}}^{an} \to Z_{\mathbf{M}}^\circ$ with $\mathbf{S} \cap Z_{\mathbf{M}}^{an}$ finite (see II.1.2). Thus one can also identify $X^*(Z_{\mathbf{M}}^\circ)$ with a finite index subgroup of $X^*(\mathbf{S} \times Z_{\mathbf{M}}^{an}) \simeq X^*(\mathbf{S}) \oplus X^*(Z_{\mathbf{M}}^{an})$. Taking the $\mathrm{Gal}(F^s/F)$-invariants, we get an injection $X^*(Z_{\mathbf{M}}^\circ)_F \hookrightarrow X^*(\mathbf{S})_F \oplus X^*(Z_{\mathbf{M}}^{an})_F \simeq X^*(\mathbf{S})_F$, where we have used [Spr98, Proposition 13.2.2 (ii)] for the last equality. Therefore, we have an injection

$$X^*(\mathbf{M})_F \hookrightarrow X^*(Z_{\mathbf{M}}^\circ)_F \hookrightarrow X^*(\mathbf{S})_F \simeq X^*(\mathbf{S}),$$

which canonically identifies $X^*(\mathbf{M})_F$ with a subgroup of finite index in $X^*(\mathbf{S})$. □

In the following lemma we will see that, the injective finite-cokernel group homomorphism $X^*(\mathbf{M})_F \hookrightarrow X^*(\mathbf{S})$ induces a unique homomorphism $\mathbf{M}(F) \to X_*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{Q}$ that extend $\nu_M$ (defined in §II.3.1).

**Lemma** II.3.2.2. *There exists a unique homomorphism $\nu_M \colon \mathbf{M}(F) \to X_*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that*

$$\langle \nu_M(z), \chi|_{\mathbf{S}} \rangle = -\omega(\chi(z)), \text{ for all } z \in \mathbf{M}(F) \text{ and } \chi \in X^*(\mathbf{M})_F (\hookrightarrow X^*(\mathbf{S})).$$

*Proof.* Lemma II.3.2.1 asserts that the restriction map $X^*(\mathbf{M})_F \to X^*(\mathbf{S})$ is injective, and has an image of finite index, i.e. for any $\chi \in X^*(\mathbf{S})$, the character $[X^*(\mathbf{S}) : X^*(\mathbf{M})_F]\chi \in X^*(\mathbf{S})$ extends to a unique $\widetilde{\chi} \in X^*(\mathbf{M})_F$. Consider the following map

$$M(F) \to \mathrm{Hom}(X^*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{Q}, \mathbb{Q}), \quad z \mapsto \left(\chi \otimes r \mapsto -r \frac{1}{[X^*(\mathbf{S}) : X^*(\mathbf{M})_F]} \omega(\widetilde{\chi}(z))\right).$$

This is clearly a homomorphism of groups. Moreover $X^*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $X_*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{Q}$ remain a dual pair of $\mathbb{Q}$-modules for the canonical pairing $\langle\, , \rangle_{\mathbb{Q}}$[19] which extend the pairing $\langle\, , \rangle \colon X_*(\mathbf{S}) \times X^*(\mathbf{S}) \to \mathbb{Z}$ (§II.1.4). Thus, any homomorphism in $\mathrm{Hom}(X^*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{Q}, \mathbb{Q})$ is of the form $\chi \otimes r \mapsto \langle \chi, \lambda \rangle r r'$ for some $\lambda \otimes r' \in X_*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{Q}$. This defines a unique homomorphism of groups $\nu_M \colon \mathbf{M}(F) \to X^*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{Q}$ verifying

$$\langle \nu_M(z), \chi \otimes 1 \rangle_{\mathbb{Q}} = -\frac{1}{[X^*(\mathbf{S}) : X^*(\mathbf{M})_F]} \omega(\widetilde{\chi}(z)) \quad \text{for all } z \in \mathbf{M}(F) \text{ and } \chi \in X^*(\mathbf{S}),$$

[19] See [Bou59, chap. IX, §1, Proposition 1].

or equivalently (since $X^*(\mathbf{M})_F$ generates $X^*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{Q}$)

$$\langle \nu_M(z), \chi_{|\mathbf{S}} \otimes 1 \rangle_{\mathbb{Q}} = -\omega(\chi(z)) \quad \text{for all } z \in \mathbf{M}(F) \text{ and } \chi \in X^*(\mathbf{M})_F. \quad \square$$

**Lemma** II.3.2.3. *The subgroup* $\mathbf{M}(F)^1 := \ker \nu_M$ *is the maximal compact open subgroup of* $\mathbf{M}(F)$.

*Proof.* See [Lan96, Proposition 1.2]. □

In addition, $\mathbf{M}(F)^1$ is a normal subgroup of $\mathbf{N}(F)$. Indeed, for any $n \in \mathbf{N}(F)$ and $z \in \mathbf{M}^1(F)$ one has[20]

$$\langle \nu_M(nzn^{-1}), \chi \rangle_{\mathbb{Q}} = \langle \nu_M(z), \chi' \rangle_{\mathbb{Q}} = 0, \quad \forall \chi \in X^*(\mathbf{M})_F,$$

where, $\chi' := (m \mapsto \chi(nmn^{-1})) \in X^*(\mathbf{M})_F$. We then have a short exact sequence of groups

$$0 \to \mathbf{M}(F)/\,\mathbf{M}(F)^1 \to \mathbf{N}(F)/\,\mathbf{M}(F)^1 \to \mathbf{N}(F)/\,\mathbf{M}(F) \to 0,$$

in which $\mathbf{M}(F)/\,\mathbf{M}(F)^1$ is a free abelian group containing $X_*(\mathbf{S})$ and having same rank [Lan96, Lemma 1.3], and $\mathbf{N}(F)/\,\mathbf{M}(F)$ is the relative Weyl group.

Consider the $\mathbb{R}$-vector space $V = X_*(\mathbf{S}\,/\,\mathbf{Z}_{c,sp}) \otimes_{\mathbb{Z}} \mathbb{R}$. We identify its dual space $V^*$ and $X^*(\mathbf{S}\,/\,\mathbf{Z}_{c,sp}) \otimes_{\mathbb{Z}} \mathbb{R}$ using the canonical extension pairing $\langle\,,\rangle_{\mathbb{R}}$ [Bou59, §1, Proposition 1]. We denote by

$$\nu \colon \mathbf{M}(F) \to V,$$

the composition of the map $\nu_M \colon \mathbf{M}(F) \to X_*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{R}$ (Lemma II.3.2.2) and the natural projection $X_*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{R} \twoheadrightarrow V$. Recall that $W = \mathbf{N}(F)/\,\mathbf{M}(F)$ acts by conjugation on $\mathbf{S}$, this induces a faithful linear action on $X_*(\mathbf{S})$. This gives a canonical homomorphism

$$j \colon W \to \mathbf{GL}(X_*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{R}),$$

which induces a homomorphism of groups $W \to \mathbf{GL}(V)$ since $\mathbf{N}(F)$ acts trivially on $\mathbf{Z}_{c,sp}$.

Let $A$ be an arbitrary affine space under $V$[21]. Using the above maps $\nu$ and $j$ we get the

[20]Recall that $\mathbf{M}(F)$ is a normal subgroup of $\mathbf{N}(F)$.

[21]Reminder: an affine space $A$ under a vector space $V$, is a set $A$ together with a simply transitive action of the vector space $V$ on it. We denote the action of any $v \in V$ on $a \in A$ by $v + a$, and refer to the map $a \mapsto v + a$ as a translation. An affine map $f \colon A \to A'$ of an affine space $A$ under $V$ into an affine space $A'$ under $V'$ consists of a map $f$ of the set $A$ to $A'$, and a linear map $d(f) \colon V \to V'$ such that $f(v + a) = d(f)(v) + f(a)$ for all $a \in A$ and all $v \in V$. The linear map $d(f)$ is called the linear part of $f$. The group of all invertible affine transformations of $A$, is called the affine group of $A$ and will be denoted $\mathrm{Aff}(A)$. We have a natural exact sequence

$$0 \to V \to \mathrm{Aff}(A) \to \mathbf{GL}(V) \to 1,$$

following diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{M}(F)/\mathbf{M}(F)^1 \arrow[r] \arrow[d, "\nu"] & \mathbf{N}(F)/\mathbf{M}(F)^1 \arrow[r] & W \arrow[r] \arrow[d, "j"] & 0 \\ 0 \arrow[r] & V \arrow[r] & \mathrm{Aff}(A) \arrow[r] & \mathbf{GL}(V) \arrow[r] & 0 \end{tikzcd}$$

Since $\mathrm{Aff}(A) \simeq V \rtimes \mathbf{GL}(V)$ [Bou62, §9, 1], putting together the actions of $\mathbf{M}(F)/\mathbf{M}(F)^1$ and $W$ on $A$, one can construct a group homomorphism $\nu_N\colon \mathbf{N}(F) \to \mathrm{Aff}(A)$ which makes the above diagram commute, i.e. for all $z \in \mathbf{M}(F)$, $\nu_N(z)$ is the translation $a \mapsto \nu(z) + a$ ($\forall a \in A$), and for any $n \in \mathbf{N}(F)$ the linear part of $\nu_N(n)$ is equal to $j(w(n))$, where $w(n)$ denotes the image of $n$ in $W$.

**Proposition** II.3.2.1. *There is a canonical affine space $\mathbb{A}_{\mathrm{red}}(\mathbf{G}, \mathbf{S})$ under $V$ (unique up to a unique isomorphism of affine spaces) together with a group homomorphism $\nu_N\colon \mathbf{N}(F) \to \mathrm{Aff}(\mathbb{A}_{\mathrm{red}}(\mathbf{G}, \mathbf{S}))$ extending $\nu$. Accordingly, we have a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{M}(F)/\mathbf{M}(F)^1 \arrow[r] \arrow[d, "\nu"] & \mathbf{N}(F)/\mathbf{M}(F)^1 \arrow[r] \arrow[d, "\nu_N"] & W \arrow[r] \arrow[d, "j"] & 0 \\ 0 \arrow[r] & V \arrow[r] & \mathrm{Aff}(\mathbb{A}_{\mathrm{red}}(\mathbf{G}, \mathbf{S})) \arrow[r] & \mathbf{GL}(V) \arrow[r] & 0 \end{tikzcd}$$

*Proof.* See the proof of [Lan96, Proposition 1.8] for the details. □

**Remark** II.3.2.1. *All possible extensions of $\nu$ are of the form*

$$\nu_{N,v}(n)\colon a \mapsto -v + \nu_N(n)(v + a),\ \forall a \in \mathbb{A}_{\mathrm{red}}(\mathbf{G}, \mathbf{S}),$$

*for some fixed $v \in V$.*

We are now ready to define the central combinatorial object of this subsection

**Definition** II.3.2.1. *The affine space $\mathbb{A}_{\mathrm{red}}(\mathbf{G}, \mathbf{S})$ together with the group homomorphism $\nu_N\colon \mathbf{N}(F) \to \mathrm{Aff}(\mathbb{A}_{\mathrm{red}}(\mathbf{G}, \mathbf{S}))$ is called the standard apartment of $\mathbf{G}$ with respect to $\mathbf{S}$.*

## II.3.3 A discrete valuation of the generating root datum

One of the consequences of Theorem II.2.7.1 is

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where, the second map is the one sending a vector $v$ to the translation $a \mapsto v + a$, and the third map is the one sending a linear map $f$ to its linear part $d(f)$.

**Proposition** II.3.3.1. *For every $\alpha \in \Phi$ and every $u \in \mathbf{U}_\alpha(F) \setminus \{1\}$ there exist a unique pair $(u', u'') \in \mathbf{U}_{-\alpha}(k) \setminus \{1\} \times \mathbf{U}_{-\alpha}(k) \setminus \{1\}$ such that $m_\alpha(u) := u'uu''$ verifies*

$$m_\alpha(u)\mathbf{U}_\alpha(k)m_\alpha(u)^{-1} = \mathbf{U}_{-\alpha}(k) \text{ and } m_\alpha(u)\mathbf{U}_{-\alpha}(k)m_\alpha(u)^{-1} = \mathbf{U}_\alpha(k).$$

*In particular,*

$$\mathbf{U}_{-\alpha}(F)u\mathbf{U}_{-\alpha}(F) \cap \mathbf{N}(F) = \{m_\alpha(u)\}.$$

*Moreover, $m_\alpha(u)$ normalizes $\mathbf{S}$ and its image in $W = \mathbf{N}(F)/\mathbf{M}(F)$ acts on $V$ by the reflexion $s_\alpha$*[22] *defined by $\alpha$ as follows*

$$s_\alpha \colon V \to V, v \mapsto -\alpha(v)\alpha^\vee + v,$$

*where, (by abuse of notation) $\alpha^\vee$ denotes the image of the co-root in $V$. Thus, the set $M_\alpha$ defined above Theorem II.2.7.1 is precisely the right coset $m_\alpha(u)\,\mathbf{M}(F)$.*

*Proof.* See [BT72, §6.1.2]. □

Fix any point (origin) $a_0$ in $\mathbb{A}_{\mathrm{red}}(\mathbf{G},\mathbf{S})$. For all $\alpha \in \Phi$ and all $u \in \mathbf{U}_\alpha(F) \setminus \{1\}$ we have[23]:

$$\nu_N(m_\alpha(u))(a) = s_\alpha(a - a_\circ) + \nu_N(m_\alpha(u))(a_\circ) = -\alpha(a - a_\circ)\alpha^\vee + a - a_\circ + \nu_N(m_\alpha(u))(a_\circ)$$

The set of fixed points of $\nu_N(m_\alpha(u))$ is an affine hyperplane in $\mathbb{A}_{\mathrm{red}}(\mathbf{G},\mathbf{S})$ with direction $\ker(s_\alpha - \mathrm{Id}_V)$. Let $b_{a_\circ,\alpha,u}$ be any element of this hyperplane, one has

$$\nu_N(m_\alpha(u))(a_\circ) = \alpha(b_{a_\circ,\alpha,u} - a_\circ)\alpha^\vee + a_\circ.$$

Therefore,

$$\nu_N(m_\alpha(u))(a) = s_\alpha(a - a_\circ) + \alpha(b_{a_\circ,\alpha,u} - a_\circ)\alpha^\vee + a_\circ.$$

Now, by setting $\varphi_\alpha^{a_\circ}(u) := -\alpha(b_{a_\circ,\alpha,u} - a_\circ) \in \mathbb{R}$ one can rewrite $\nu_N(m_\alpha(u))$ as follows:

$$\nu_N(m_\alpha(u))(a) = -\left(\alpha(a - a_\circ) + \varphi_\alpha^{a_\circ}(u)\right)\alpha^\vee + a.$$

Accordingly, the element $\nu_N(m_\alpha(u))$ acts then as a reflection at the hyperplane $\{a \in \mathbb{A}_{\mathrm{red}}(\mathbf{G},\mathbf{S}) \colon \alpha(a - a_\circ) = -\varphi_\alpha^{a_\circ}(u)\}$ (See [BT72, Remark (b) 6.2.12]).

One of the fundamental results obtained by Bruhat and Tits is

**Theorem** II.3.3.1. *The family $\varphi^{a_\circ} = (\varphi_\alpha^{a_\circ} \colon \mathbf{U}_\alpha(F) \to \mathbb{R} \cup \{\infty\})_{\alpha\in\Phi}$ is a discrete valuation of the generating root datum $\left(\mathbf{M}(F), (\mathbf{U}_\alpha(F), M_\alpha)_{\alpha\in\Phi}\right)$ (see Theorem II.2.7.1), meaning that $\varphi^{a_\circ}$ has the following properties [BT72, 6.2.1 & 6.2.21]:*

[22] Or equivalently, the linear part of the affine action of $m_\alpha(u)$ is $s_\alpha$.

[23] since $\nu_N(m_\alpha(u))$ is an affine map with linear part equal to the reflection $s_\alpha$, we have for all $a \in \mathbb{A}_{\mathrm{red}}(\mathbf{G},\mathbf{S})$, see footnote 21 on page 42.

*(V 0) For all $\alpha \in \Phi$ the image of $\varphi_\alpha^{a_\circ}$ contains at least 3 elements.*

*(V 1) For all $\alpha \in \Phi$ and $r \in \mathbb{R} \cup \{\infty\}$, the set $U_{\alpha+r}^{a_\circ} := (\varphi_\alpha^{a_\circ})^{-1}([r, \infty])$ is a compact open subgroup of $\mathbf{U}_\alpha(F)$ and $U_{\alpha+\infty}^{a_\circ} = \{1\}$.*

*(V 2) For all $\alpha \in \Phi$ and all $n \in M_\alpha$, the function $u \mapsto \varphi_{-\alpha}^{a_\circ}(u) - \varphi_\alpha^{a_\circ}(nun^{-1})$ is constant on $\mathbf{U}_{-\alpha}(F) \setminus \{1\}$.*

*(V 3) For any $\alpha, \beta \in \Phi$ and $r, r' \in \mathbb{R}$ such that $\beta \notin -\mathbb{R}_+\alpha$, the group of commutators $[U_{\alpha+r}^{a_\circ}, U_{\beta+r'}^{a_\circ}]$ lies in the group generated by the groups $U_{p\alpha+q\beta+pr+qr'}$ where $p, q \in \mathbb{Z}_{>0}$ and $p\alpha + q\beta \in \Phi$.*

*(V 4) If both $\alpha$ and $2\alpha$ belong to $\Phi$, the restriction of $2\varphi_\alpha^{a_\circ}$ to $U_{2\alpha}^{a_\circ}$ is equal to $\varphi_{2\alpha}^{a_\circ}$.*

*(V 5) Let $\alpha \in \Phi$, $u \in \mathbf{U}_\alpha(F)$ and $u', u'' \in \mathbf{U}_{-\alpha}(F)$; if $u'uu'' \in M_\alpha$ then $\varphi_{-\alpha}^{a_\circ}(u') = -\varphi_\alpha^{a_\circ}(u)$.*

*Proof.* For a proof we refer the reader to [BT84, 5.1.23] and [BT72, 6.2.12 (b)]. □

For every $v \in V$, the map $a_\circ \mapsto v + a_\circ$ yields the map

$$\varphi^{a_\circ} \mapsto v + \varphi^{a_\circ} := \varphi^{v+a_\circ} = \left(\varphi_\alpha^{v+a_\circ} \colon \mathbf{U}_\alpha(F) \to \mathbb{R} \cup \{\infty\}, u \mapsto \varphi^{a_\circ}(u) + \alpha(v)\right)_{\alpha \in \Phi}.$$

Two discrete valuations in $\mathbb{A}_{\varphi^{a_\circ}} := \{v + \varphi^{a_\circ} \colon v \in V\}$ will be said equipollent. As noted by [BT72, 6.2.6], the action of $V$ on $\mathbb{A}_{\varphi^{a_\circ}}$ described above endow this latter with a structure of an affine space under $V$, which we endow with the euclidean distance corresponding to the scalar product given on $V$.

**Remark** II.3.3.1. *Under the isomorphism of affine spaces $\mathbb{A}_{\varphi^{a_\circ}} \simeq \mathbb{A}_{\mathrm{red}}(\mathbf{G}, \mathbf{S})$, the action of $N$ on $\mathbb{A}_{\mathrm{red}}(\mathbf{G}, \mathbf{S})$ described in Lemma II.3.2.1 correspond to the following action of $N$ on $\mathbb{A}_{\varphi^{a_\circ}}$:*

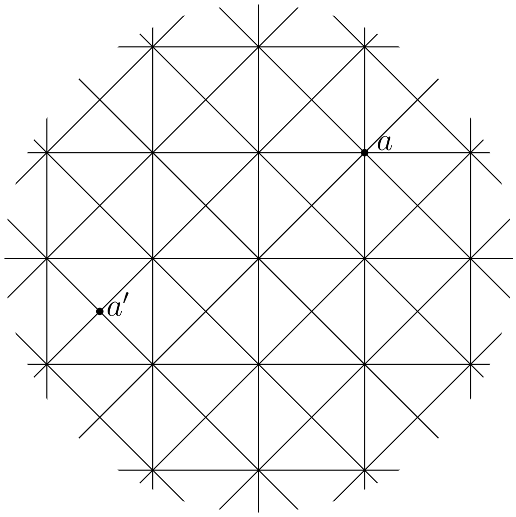
$$N \times \mathbb{A}_{\varphi^{a_\circ}} \to \mathbb{A}_{\varphi^{a_\circ}}, \quad [n, v + \varphi^{a_\circ}] \mapsto w(n)(v) + \varphi^{\nu_N(n)(a_\circ)},$$

*where $w(n)$ denotes the image of $n \in N$ in $W$.*

**Definition** II.3.3.1. *We will say that a discrete valuation $\varphi^a \in \mathbb{A}_{\varphi^{a_\circ}}$ (for some $a \in \mathbb{A}_{\mathrm{red}}(\mathbf{G}, \mathbf{S})$) is special if, for any $\alpha \in \Phi_{\mathrm{red}}$, one has $0 \in \Gamma_\alpha(a)$.*

**Remark** II.3.3.2. *By definition $\varphi^a = (\varphi_\alpha^a \colon u \mapsto -\alpha(b_{a,\alpha,u} - a))$, we see then that $\varphi^a$ is special if and only if for any $\alpha \in \Phi_{\mathrm{red}}$, there is a $u \in \mathbf{U}(F)$ such that $a = b_{a,\alpha,u}$, this is equivalent to say that $a$ is fixed by $m_\alpha(u)$, in which case we will say that $a$ is a special point of $\mathbb{A}_{\mathrm{red}}(\mathbf{G}, \mathbf{S})$.*

**Remark** II.3.3.3. *Here is a "pictural" equivalent definition for being special: Any vertex of the apartment $\mathbb{A}_{\mathrm{red}}(\mathbf{G}, \mathbf{S})$ is the intersection of at least $\dim_{\mathbb{R}} V$ different reflection* *hyperplanes. A vertex is called special, if the set of reflection hyperplanes that contains it, meets every parallelism class of reflection hyperplanes of $\mathbb{A}_{\mathrm{red}}(\mathbf{G},\mathbf{S})$. With this alternative definition, one can see that in the following figure*[24] *"$a$" is special while "$a'$" is not.*




□

**Lemma** II.3.3.1. *There exists a special discrete valuation in $\mathbb{A}_{\varphi^{a_\circ}}$.*

*Proof.* This is [BT72, Corollaire 6.2.15]. □

By the above lemma we may and will assume from now on that $a_\circ$ is a special point. We will also omit indicating $a_\circ$ in the notation of $\varphi^{a_\circ}, \varphi_\alpha^{a_\circ}$ and $U_{\alpha+r}^{a_\circ}$ for $\alpha \in \Phi, r \in \mathbb{R}$, i.e. we will write $\varphi, \varphi_\alpha$ and $U_{\alpha+r}$. Set

$$\Gamma_\alpha := \{\varphi_\alpha(u) \colon u \in \mathbf{U}_\alpha(F) \setminus \{1\}\},$$

$$\Gamma'_\alpha := \{\varphi_\alpha(u) \colon u \in \mathbf{U}_\alpha(F) \setminus \{1\} \text{ and } \varphi_\alpha(u) = \sup \varphi_\alpha(u\mathbf{U}_{2\alpha}(F))\}\,.$$

**Lemma** II.3.3.2. *For all $\alpha \in \Phi$, we have: $\Gamma_\alpha = \Gamma_{-\alpha}$ and $\Gamma_\alpha = \Gamma'_\alpha \cup \frac{1}{2}\Gamma_{2\alpha}$, in particular $\Gamma_\alpha = \Gamma'_\alpha$ if $2\alpha \notin \Phi$.*

*Proof.* See [BT72, 6.2.2]. □

**Proposition** II.3.3.2. *For every $\alpha \in \Phi_{\mathrm{red}}$, there exists a positive integer $n_\alpha$ such that $\Gamma_\alpha = n_\alpha^{-1}\mathbb{Z}$, satisfying the following properties: $n_{w\alpha} = n_\alpha$ for every $w \in W = W(\Phi)$, and $n_{2\alpha} \in \left\{\frac{1}{2}n_\alpha, n_\alpha\right\}$ if $\alpha, 2\alpha \in \Phi$.*

*Proof.* This follows from [SS97, Lemma I.2.10] and [BT72, 6.2.23]. □

[24]Which depicts roughly the case where the initial root system is of type $C_2$.

## II.3.4 Affine roots

**Definition** II.3.4.1. *Define for every $\alpha \in \Phi_{\mathrm{red}}$ and every $r \in \Gamma_\alpha$ the affine mapping $\alpha + r\colon \mathbb{A}_{\mathrm{red}}(\mathbf{G}, \mathbf{S}) \to \mathbb{R}$ given by $a \mapsto \alpha(a - a_\circ) + r$. Let $\Phi_{\mathrm{aff}}$ denote the set of all affine roots $\{\alpha + r\colon \alpha \in \Phi_{\mathrm{red}}, r \in \Gamma'_\alpha\}$ [BT72, 6.2.6].*

For use in Chapter III, we describe in the following lemma the action of $N$ on root groups $U_{\alpha+r}$ for any affine root $\alpha + r$.

**Lemma** II.3.4.1. *Let $n \in N$, with image $w$ in $W$. We have for all $\alpha \in \Phi_{\mathrm{red}}$ and all $r \in \Gamma_\alpha$:*

$$nU_{\alpha+r}n^{-1} = U_\beta,$$

*where, $\beta = w(\alpha) + r - w(\alpha)(\nu_N(n)(a_\circ) - a_\circ)$.*

In particular, if $n = m \in \mathbf{M}(F)$ then

$$mU_{\alpha+r}m^{-1} = U_{\alpha+r-\langle\nu(m),\alpha\rangle}.$$

*Proof.* Let $\alpha + r \in \Phi_{\mathrm{aff}}$, and $n \in N$ of image $w$ in $W$. As suggested by the proof of [BT72, 6.2.10. (iii)], we have

$$\begin{aligned}
nU_{\alpha+r}n^{-1} &= (\varphi_\alpha)^{-1}([r, \infty]) \\
&= n\{u \in \mathbf{U}_\alpha(F)\colon \varphi_\alpha(u) \geq r\}n^{-1} \\
&= \{u \in \mathbf{U}_{w(\alpha)}(F)\colon \varphi_\alpha(n^{-1}un) \geq r\} \\
&= \{u \in \mathbf{U}_{w(\alpha)}(F)\colon (\nu_N(\varphi))_{w(\alpha)}(u) \geq r\} \\
&\overset{\text{Remark II.3.3.1}}{=} \{u \in \mathbf{U}_{w(\alpha)}(F)\colon (w(0_V) + \varphi^{\nu_N(a_\circ)})_{w(\alpha)}(u) \geq r\} \\
&= \{u \in \mathbf{U}_{w(\alpha)}(F)\colon (\varphi^{\nu_N(a_\circ)})_{w(\alpha)}(u) \geq r\} \\
&\overset{\varphi^{v+a_\circ}=v+\varphi^{a_\circ}}{=} \{u \in \mathbf{U}_{w(\alpha)}(F)\colon (\nu_N(n)(a_\circ) - a_\circ + \varphi)_{w(\alpha)}(u) \geq r\} \\
&= \{u \in \mathbf{U}_{w(\alpha)}(F)\colon \varphi_{w(\alpha)}(u) + w(\alpha)(\nu_N(n)(a_\circ) - a_\circ) \geq r\} \\
&= U_\beta
\end{aligned}$$

where, $\beta = w(\alpha) + r - w(\alpha)(\nu_N(n)(a_\circ) - a_\circ)$. $\square$

In a similar way to the non-affine case, consider for each affine root $\alpha + r \in \Phi_{\mathrm{aff}}$ the affine hyperplanes

$$H_{\alpha+r} := \big\{a \in \mathbb{A}_{\mathrm{red}}(\mathbf{G}, \mathbf{S})\colon \alpha(a - a_\circ) = -r\big\}.$$

For any $u_r \in \varphi_\alpha^{-1}(r)$, one has

$$H_{\alpha+r} = \{a \in \mathbb{A}_{\mathrm{red}}(\mathbf{G}, \mathbf{S})\colon m_\alpha(u_r)(a) = a\}.$$

These hyperplanes define a poly-simplicial structure on the standard apartment in the following way:

**Definition** II.3.4.2. *Define the equivalence relation on $\mathbb{A}_{\mathrm{red}}(\mathbf{G}, \mathbf{S})$ by $a \sim b$ if for every affine root $\beta$ the sign of $\beta(a)$ and $\beta(b)$ is the same or are both equal to zero* [25]*; the equivalence classes are called the facets. A vertex is a point which is a facet, e.g. the point $a_\circ$. A facet with maximal dimension is called an alcove, it is also a connected component $\mathbb{A}_{\mathrm{red}}(\mathbf{G}, \mathbf{S}) \setminus \bigcup_{\beta \in \Phi_{\mathrm{aff}}} H_\beta$.*

From now on, we fix an alcove $\mathfrak{a} \subset \mathbb{A}_{\mathrm{red}}(\mathbf{G}, \mathbf{S})$ containing in its closure the special vertex $a_\circ$.

**Definition** II.3.4.3. *Let $\alpha \in \Phi$ and $r \in \Gamma_\alpha$. We say that $\alpha + r$ is $\mathfrak{a}$-positive (resp. $\mathfrak{a}$-negative) if $\alpha(a - a_\circ) + r > 0$ (resp. $< 0$) for some $a \in \mathfrak{a}$ (then for all, since the sign does not depend on the choice of $a \in \mathfrak{a}$). Let $\Phi_{\mathrm{aff}}^+$ (resp. $\Phi_{\mathrm{aff}}^-$) denote the set of affine $\mathfrak{a}$-positive (resp. negative) affine roots.*

For any non-empty subset $\Omega \subset \mathbb{A}_{\mathrm{red}}(\mathbf{G}, \mathbf{S})$ and $\alpha \in \Phi$, we define $f_\Omega\colon \Phi \to \mathbb{R} \cup \{\infty\}$,

$$f_\Omega(\alpha) := \inf\{r \in \Gamma_\alpha\colon (\alpha + r)(\Omega) \subset \mathbb{R}^+\}$$

Define also the subgroup $U_\Omega$ to be the group of $\mathbf{G}(F)$ generated by $\cup_{\alpha \in \Phi_{\mathrm{red}}} U_{\alpha + f_\Omega(\alpha)}$. Note that for $a \in \mathbb{A}_{\mathrm{red}}(\mathbf{G}, \mathbf{S})$ and $\alpha \in \Phi$, the real $f_{\{a\}}(\alpha)$ depends only on the facet containing $a$.

**Example** II.3.4.1. *For $\Omega = \{a_\circ\}$, we have $f_{\{a_\circ\}}(\alpha) = 0$ for all $\alpha \in \Phi$. Now, let us see the case $\Omega = \mathfrak{a}$. We first have*

$$\mathfrak{a} = \{a \in \mathbb{A}_{\mathrm{red}}(\mathbf{G}, \mathbf{S})\colon 0 < \alpha(a - a_\circ) < n_\alpha^{-1} \text{ for all } \alpha \in \Phi_{\mathrm{red}}\ \mathfrak{a}\text{-positive}\}.$$

*Therefore, if $\alpha \in \Phi$ is $\mathfrak{a}$-positive, then $f_{\mathfrak{a}}(\alpha) = 0$. If now $\alpha \in \Phi_{\mathrm{red}}$ is $\mathfrak{a}$-negative, then $-\alpha$ is $\mathfrak{a}$-positive, hence for all $a \in \mathfrak{a}$ we have $0 < -\alpha(a - a_\circ) < n_{-\alpha}^{-1}$. The real $f_{\mathfrak{a}}(\alpha) \in \Gamma_\alpha$ being the smallest element such $\alpha(a - a_\circ) \geq -f_{\mathfrak{a}}(\alpha)$, we see that $f_{\mathfrak{a}}(\alpha) = n_{-\alpha}^{-1}$. By Lemma*

[25] Equivalently, $a \sim b$ if for any affine hyperplane $H$, either $a, b \in H$ or they are in the same connected component of $\mathbb{A}_{\mathrm{red}}(\mathbf{G}, \mathbf{S}) \setminus H$.

*II.3.3.2, we have $\Gamma_\alpha = \Gamma_{-\alpha}$, this implies that $n_\alpha^{-1} = n_{-\alpha}^{-1}$. In conclusion:*

$$f_{\mathfrak{a}}(\alpha) = \begin{cases} 0 & \text{if } \alpha \in \Phi \text{ is } \mathfrak{a}\text{-positive,} \\ n_\alpha^{-1} & \text{if } \alpha \in \Phi_{\mathrm{red}} \text{ is } \mathfrak{a}\text{-negative.} \end{cases}$$

**Proposition** II.3.4.1. *For any non-empty subset $\Omega \subset \mathbb{A}_{\mathrm{red}}(\mathbf{G}, \mathbf{S})$, the groups defined above have the following important properties:*

1. *For any $n \in \mathbf{N}(F)$ we have $nU_\Omega n^{-1} = U_{\nu(n)\cdot\Omega}$, so in particular $N_\Omega := \{n \in \mathbf{N}(F) \colon \nu(n) \cdot x = x \text{ for all } x \in \Omega\}$ normalizes $U_\Omega$.*

2. *For any $\alpha \in \Phi_{\mathrm{red}}$ we have $U_\Omega \cap U_\alpha = U_{\alpha+f_\Omega(\alpha)}$.*

3. *The set $P_\Omega := N_\Omega U_\Omega = U_\Omega N_\Omega$ is a group. We have $P_\Omega \cap \mathbf{N}(F) = N_\Omega$:*

   *For any decomposition into positive and negative roots $\Phi = \Phi^+ \sqcup \Phi^-$, we have*

4. *The following product map is an*

$$\textstyle\prod_{\alpha\in\Phi_{\mathrm{red}}\cap\Phi^\pm} U_{\alpha+f_\Omega(\alpha)} \xrightarrow{\sim} U_\Omega \cap \mathbf{U}(F)^\pm =: U_\Omega^\pm.$$

   *homeomorphism whatever ordering of the factors we take.*

5. *$U_\Omega = U_\Omega^+ U_\Omega^- (U_\Omega \cap \mathbf{N}(F))$.*

*Proof.* See [BT72, 6.2.10(iii),6.4.9 & 7.1.3]. □

From now, we will adopt the following notation: when $\Omega = \{x\}$ we will write $\square_x$ instead of $\square_{\{x\}}$ for $\square \in \{f, U, N, P\}$.

### II.3.5 Affine Weyl groups

For every affine root $\beta = \alpha + r \in \Phi_{\mathrm{aff}}$, let $s_\beta \in \nu_N(\mathbf{N}(F))$ denote the orthogonal reflection with respect to the hyperplane $H_\beta$:

$$s_\beta \colon a \mapsto -\left(\alpha(a - a_\circ) + r\right)\alpha^\vee + a, \ \forall a \in \mathbb{A}_{\mathrm{red}}(\mathbf{G}, \mathbf{S}).$$

**Definition** II.3.5.1. *We define the affine Weyl group $W_{\mathrm{aff}} \subset \nu_N(\mathbf{N}(F))$ to be the group generated by the reflections $s_\beta$ for all $\beta \in \Phi_{\mathrm{aff}}$. It is normal a subgroup of $\nu_N(\mathbf{N}(F))$ [BT72, 6.2.11].*

The set $\Phi' := \{\alpha \in \Phi \colon \Gamma'_\alpha \neq 0\}$ is a root system that contains $\Phi_{nm} = \{\alpha \in \Phi \colon 2\alpha \notin \Phi\}$ [BT72, 6.2.22]. Define the root system $\Sigma := \{n_\alpha \alpha \colon \alpha \in \Phi'\}$ and consider the following "échelonnage"[26]

$$\mathcal{E} = \{(\alpha, n_\alpha \alpha) \colon \alpha \in \Phi'\} \subset \Phi' \times \Sigma.$$

Set $\Sigma_{\mathrm{aff}} = \cup_{\alpha \in \Sigma}(\alpha + \mathbb{Z})$. The map

$$\Phi_{\mathrm{aff}} = \cup_{\alpha \in \Phi_{\mathrm{red}}}(\alpha + \Gamma_\alpha) \longrightarrow \Sigma_{\mathrm{aff}}; \quad (\alpha + r) \longmapsto n_\alpha(\alpha + r),$$

is bijective and respects positivity [Vig16, (38)]. The group $W_{\mathrm{aff}} = W_{\mathrm{aff}}(\Sigma)$ is the affine Weyl group associated to $\Sigma$ [Bou68, VI §2.1 & §2.5 Proposition 8] and [BT72, 6.2.22]. Define $\Lambda_{\mathrm{aff}}$ to be the subgroup of translations in $W_{\mathrm{aff}}$, we can then identify $\Lambda_{\mathrm{aff}}$ with the $\mathbb{Z}$-module generated by the set $\Sigma^\vee$ of coroots [Bou68, VI §2.1]. The affine Weyl group $W_{\mathrm{aff}}$ acts simply transitively on the set of alcoves in $\mathbb{A}_{\mathrm{red}}(\mathbf{G}, \mathbf{S})$ [Bou68, VI 2.1].

Let $a$ be vertex in $\mathbb{A}_{\mathrm{red}}(\mathbf{G}, \mathbf{S})$, we denote by $\Phi^a_{\mathrm{aff}}$ the set of affine roots that vanish at $a$, set $W^a_{\mathrm{aff}} = \langle s_\beta \colon \beta \in \Phi^a_{\mathrm{aff}} \rangle \subset W_{\mathrm{aff}}$. The vertex $a$ is special in the sense of Remark II.3.3.2 if and only if the composition of the following maps

$$W^a_{\mathrm{aff}} \hookrightarrow W_{\mathrm{aff}} \hookrightarrow \nu_N(\mathbf{N}(F)) \twoheadrightarrow W,$$

is an isomorphism[27]. Special vertices exists by [Bou68, V §3.10 Proposition 10]. We have by [BT72, 1.3 & 6.2.19] a decomposition for the affine Weyl group

$$W_{\mathrm{aff}} \simeq \Lambda_{\mathrm{aff}} \rtimes W^a_{\mathrm{aff}} \simeq \mathbb{Z}[\Sigma^\vee] \rtimes W^a_{\mathrm{aff}},$$

for any fixed special vertex $a \in \mathbb{A}_{\mathrm{red}}(\mathbf{G}, \mathbf{S})$.

**Definition** II.3.5.2. *A face of the alcove $\mathfrak{a}$ is a facet contained in a single affine hyperplane. A wall of the alcove $\mathfrak{a}$ is a hyperplane containing a face of $\mathfrak{a}$. For every facet $\mathcal{F} \subset \mathfrak{a}$ we define its type to be the set*

$$\mathcal{T}_{\mathcal{F}} = \{s_H \colon H \text{ a wall of } \mathfrak{a},\ \mathcal{F} \subset H\}.$$

*Let $W^{\mathcal{F}}_{\mathrm{aff}}$ be the group generated by $\mathcal{T}_{\mathcal{F}}$. Then $(W^{\mathcal{F}}_{\mathrm{aff}}, \mathcal{T}_{\mathcal{F}})$ is a finite Coxeter system.*

Since a facet of $\mathcal{F}$ is the image by an element of $W_{\mathrm{aff}}$ of a unique facet of $\mathfrak{a}$, and a facet $\mathcal{F}$ of $\mathfrak{a}$ is determined by its type, one can then define the type for all facets [BT72, 1.3.5]. So alcoves have empty types ($\mathcal{T}_{\mathfrak{a}} = \emptyset$) and special points have full type ($\mathcal{T}_{a_\circ} = \{s_\beta \colon \beta \in \Phi^{a_\circ}_{\mathrm{aff}}\} \simeq \Phi^+_{\mathrm{red}}$).

[26] in the sense of [BT72, 1.4.1].
[27] This is equivalent to $W^a_{\mathrm{aff}} \simeq W$ or just to $\# W^a_{\mathrm{aff}} = \# W$ since the induced map $W^a_{\mathrm{aff}} \hookrightarrow W_{\mathrm{aff}} \to W$ is always injective.

## II.3.6 The reduced Bruhat–Tits building

We arrive now at the most important object of this subsection, that is the reduced Bruhat–Tits building:

**Definition** II.3.6.1. *Define the set*

$$\mathcal{B}(\mathbf{G},F)_{\mathrm{red}} := \mathbf{G}(F) \times \mathbb{A}_{\mathrm{red}}(\mathbf{G},\mathbf{S})/\sim,$$

*where* $\sim$ *is the equivalence relation on* $\mathbf{G}(F) \times \mathbb{A}_{\mathrm{red}}(\mathbf{G},\mathbf{S})$ *defined by*

$$(g,x) \sim (h,y) \ \text{ if } \exists n \in \mathbf{N}(F) \text{ such that } nx = y \text{ and } g^{-1}hn \in U_x.$$

The group $\mathbf{G}(F)$ acts on $\mathcal{B}(\mathbf{G},F)_{\mathrm{red}}$ on the left:

$$g \cdot [(h,y)] := [(gh,y)] \quad \text{for } g \in \mathbf{G}(F) \text{ and } (h,y) \in \mathbf{G}(F) \times \mathbb{A}_{\mathrm{red}}(\mathbf{G},\mathbf{S}).$$

Moreover, the map $\mathbb{A}_{\mathrm{red}}(\mathbf{G},\mathbf{S}) \to \mathcal{B}(\mathbf{G},F)_{\mathrm{red}}$ given by $x \mapsto [(1,x)]$ is an $\mathbf{N}(F)$-equivariant embedding. We will denote its image by $\mathcal{A}$. We have $g \cdot \mathcal{A} = \mathcal{A}$ (resp. $g \cdot x = x$, $\forall x \in \mathcal{A}$) if and only if $g \in \mathbf{N}(F)$ (resp. $g \in \ker \nu_N$, which contains the maximal compact subgroup $\mathbf{M}(F)^1$ of $\mathbf{M}(F)$ and the center of $\mathbf{G}(F)$.) [BT72, 7.4.10].

**Definition** II.3.6.2. *An apartment of the building* $\mathcal{B}(\mathbf{G},F)_{\mathrm{red}}$ *is a subset of the form* $g\mathcal{A}$ *for some* $g \in \mathbf{G}(F)$*. A facet (resp. alcove) of* $\mathcal{B}(\mathbf{G},F)_{\mathrm{red}}$ *is a subset of the form* $g\mathcal{F}$ *for some* $g \in \mathbf{G}(F)$ *and a facet (resp. alcove)* $\mathcal{F} \subset \mathcal{A}$*.*

Thanks to Theorem II.2.1.2, apartments of $\mathcal{B}(\mathbf{G},F)_{\mathrm{red}}$ are in bijection with maximal split tori of $\mathbf{G}$.

**Proposition** II.3.6.1. *In this proposition, we will collect some important properties regarding the building* $\mathcal{B}(\mathbf{G},F)_{\mathrm{red}}$*:*

1. *(Fixators) Let* $\Omega \subset \mathcal{A}$*. We have an alternative characterization for the subgroups* $P_\Omega = N_\Omega U_\Omega$ *(defined in Proposition II.3.4.1 - 3):*

   $$P_\Omega = \{g \in \mathbf{G}(F) \colon gx = x \quad \forall x \in \Omega\},$$

   *in other words* $P_\Omega$ *is the subgroup that fixes every point of* $\Omega$ *[BT72, 7.4.4]. The fixator* $P_\Omega$ *is the semidirect product [BT72, 4.1.1, 6.4.2, 7.1.3] :*

   $$P_\Omega = U_\Omega \rtimes \ker \nu.$$

2. *For any $g \in \mathbf{G}(F)$ we can find a $n \in \mathbf{N}(F)$ verifying $gx = nx$ for all $x$ in the closed subset $\mathcal{A} \cap g^{-1}\mathcal{A}$ [BT72, 7.4.8].*

3. *(Transitivity) Let $\Omega \subset \mathcal{A}$. The group $U_\Omega$ acts transitively on the set of all apartments containing $\Omega$ [BT72, 7.4.9].*

4. *For any two facets in the building, there exists an apartment that contains both of them [BT72, 7.4.18].*

5. *Let us fix an $W$-invariant euclidean metric $d$ on $\mathcal{A}$. The previous properties ensure that the distance $d$ extends uniquely to a $\mathbf{G}(F)$-invariant metric on the set $(\mathcal{B}(\mathbf{G}, F)_{\mathrm{red}}, d)$.*

**Remark** II.3.6.1. *The term "fixators" refers to pointwise stabilizers, in contrast the term "stabilizers" will be used for setwise stabilizers.* □

**Definition** II.3.6.3. *The reduced Bruhat–Tits building of $\mathbf{G}(F)$ is the pair $(\mathcal{B}(\mathbf{G}, F)_{\mathrm{red}}, d)$ together with its isometric $\mathbf{G}(F)$-action and the poly-simplicial structure defined by its facets.*

### II.3.7 The extended Bruhat–Tits building

When the center $Z(\mathbf{G})$ has split rank $> 0$, the stabilizer $P_x$ of a point $x \in \mathcal{A} \subset \mathcal{B}(\mathbf{G}, F)_{\mathrm{red}}$ is no longer a compact subgroup of $\mathbf{G}(F)$. To remedy this issue we define in this section a larger building following [BT84, 4.2.16].

We have a decomposition

$$X_*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{R} = (X_*(\mathbf{S}^{der}) \otimes_{\mathbb{Z}} \mathbb{R}) \oplus X_*(\mathbf{Z}_c) \otimes_{\mathbb{Z}} \mathbb{R}),$$

where $\mathbf{S}^{der} = (\mathbf{G}^{\mathrm{der}} \cap \mathbf{S})^\circ$. This is the "dual" statement of the decomposition (3) in Lemma II.2.2.1. This decomposition allows us to inject $\Phi^\vee$ in $V^\vee$ and again the "dual" statement of (4) in Lemma II.2.2.1 together with Theorem II.2.2.1 says that

$$V = \mathrm{span}_{\mathbb{R}}(\Phi^\vee) = X_*(\mathbf{S}^{der}) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Hence,

$$X_*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{R} = V \oplus (X_*(\mathbf{Z}_{c,sp}) \otimes_{\mathbb{Z}} \mathbb{R}).$$

This allows us to see the apartment $\mathcal{A} \subset \mathcal{B}(\mathbf{G}, F)_{\mathrm{red}}$ corresponding to $\mathbf{S}$ as a torsor for the real vector space

$$X_*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{R} / X_*(\mathbf{Z}_{c,sp}) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Let us construct a homomorphism $\nu_G \colon \mathbf{G}(F) \to X_*(\mathbf{Z}_{c,sp}) \otimes_{\mathbb{Z}} \mathbb{Q}$: Using the isogeny $\mathbf{Z}_c \times \mathbf{G}^{\mathrm{der}} \to \mathbf{G}$, we see that $X^*(\mathbf{G})_F = X^*(\mathbf{G}/\mathbf{G}^{\mathrm{der}})_F$. Since the quotient map $\mathbf{Z}_c \to \mathbf{G}/\mathbf{G}^{\mathrm{der}}$ is an $F$-isogeny of $F$-tori, we see that $X^*(\mathbf{G}/\mathbf{G}^{\mathrm{der}})_F$ identifies with a subgroup of $X^*(\mathbf{Z}_c)_F$ of finite index. Therefore, we have a natural isomorphism of $\mathbb{Q}$-vector spaces

$$X^*(\mathbf{G}/\mathbf{G}^{\mathrm{der}})_F \otimes_{\mathbb{Z}} \mathbb{Q} \to X^*(\mathbf{Z}_c)_F \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Let $\mathbf{Z}_c^{an}$ be the maximal $F$-anisotropic subtorus of $\mathbf{Z}_c$, we then have an isogeny $\mathbf{Z}_{c,sp} \times \mathbf{Z}_c^{an} \to \mathbf{Z}_c$ with $\mathbf{Z}_{c,sp} \cap \mathbf{Z}_c^{an}$ finite (see II.1.2). Thus, one has again an injection from $X^*(\mathbf{Z}_c)_F$ into $X^*(\mathbf{Z}_{c,sp})_F = X^*(\mathbf{Z}_{c,sp})$ that identifies the former group with a finite index subgroup of the latter. We then have a natural isomoprhism of $\mathbb{Q}$-vector spaces between $X^*(\mathbf{Z}_c)_F \otimes_{\mathbb{Z}} \mathbb{Q}$ and $X^*(\mathbf{Z}_{c,sp}) \otimes_{\mathbb{Z}} \mathbb{Q}$. Identify the $\mathbb{Q}$-linear dual $\mathrm{Hom}_{\mathbb{Z}}(X^*(\mathbf{Z}_{c,sp}), \mathbb{Q})$ of $X^*(\mathbf{Z}_{c,sp}) \otimes_{\mathbb{Z}} \mathbb{Q}$ with $X_*(\mathbf{Z}_{c,sp}) \otimes_{\mathbb{Z}} \mathbb{Q}$. In conclusion, we get a natural isomorphism

$$X^*(\mathbf{G})_F \otimes_{\mathbb{Z}} \mathbb{Q} \simeq X^*(\mathbf{Z}_{c,sp}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Consequently, we obtain a canonical isomorphism

$$X_*(\mathbf{Z}_{c,sp}) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathrm{Hom}_{\mathbb{Z}}(X^*(\mathbf{G})_F \otimes_{\mathbb{Z}} \mathbb{Q}, \mathbb{Q}).$$

This shows that there exists a unique homomorphism

$$\mathbf{G}(F) \to X_*(\mathbf{Z}_{c,sp}) \otimes_{\mathbb{Z}} \mathbb{Q},$$

extending $\nu_G \colon \mathbf{G}(F) \to \mathrm{Hom}_{\mathbb{Z}}(X^*(\mathbf{G})_F, \mathbb{Z})$ (§II.3.1); this is the unique homomorphism denoted by abuse of notation $\nu_G$ and satisfying[28] (see proof of Lemma II.3.2.2)

$$\langle \nu_G(g), \chi|_{\mathbf{Z}_{c,sp}} \rangle = -\omega(\chi(g)), \text{ for all } \chi \in X^*(\mathbf{G})_F \text{ and } g \in \mathbf{G}(F).$$

Put $\mathbf{G}(F)^1 := \ker \nu_G$[29]. Then $\mathbf{G}(F)/\mathbf{G}(F)^1$ is a finitely generated abelian group, and there is an isomorphism

$$\mathbf{G}(F)/\mathbf{G}(F)^1 \otimes_{\mathbb{Z}} \mathbb{Q} \simeq X_*(\mathbf{Z}_{c,sp}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Set $V_G := X_*(\mathbf{Z}_{c,sp}) \otimes_{\mathbb{Z}} \mathbb{R}$, and let $\mathbb{A}_G$ be a fixed affine space under $V_G$. We have a morphism $\nu_G \colon \mathbf{G}(F) \to \mathrm{Aff}(\mathbb{A}_G)$ sending every $g$ to the translation $(a \mapsto \nu_G(g) + a)$.

We define the extended standard apartment $\mathbb{A}_{\mathrm{ext}}(\mathbf{G}, \mathbf{S})$ to be the product of $\mathbb{A}(\mathbf{G}, \mathbf{S})_{\mathrm{red}} \times \mathbb{A}_G$

[28] We abuse notation, and denote also by $\langle\,,\rangle$ the pairing $X_*(\mathbf{Z}_{c,sp}) \times X^*(\mathbf{Z}_{c,sp}) \to \mathbb{Z}$.

[29] We refer the reader to §II.3.9.2 for more on the homomorphism $\nu_G$

together with the group homomorphism

$$\nu_{N,\text{ext}}\colon\ \mathbf{N}(F) \longrightarrow \text{Aff}(\mathbb{A}_{\text{ext}}(\mathbf{G},\mathbf{S})), \qquad n \longmapsto \nu_N(n) \oplus \nu_G(n).$$

The decomposition $X_*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{R} = V \oplus V_G$ shows that the extended standard apartment $\mathbb{A}_{\text{ext}}(\mathbf{G},\mathbf{S})$ (as defined above) is actually an affine space under $X_*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{R}$, and the restriction of $\nu_{N,\text{ext}}$ to $\mathbf{M}(F)$ corresponds precisely to the translation action given by the homomorphism $\nu_M$ of Lemma II.3.2.2.

**Remark** II.3.7.1. *While the reduced system $(\mathbb{A}(\mathbf{G},\mathbf{S})_{\text{red}}, \nu_N)$ is canonical, that is, unique up to unique isomorphism (Proposition II.3.2.1), the extended system $(\mathbb{A}_{\text{ext}}(\mathbf{G},\mathbf{S}), \nu_{N,\text{ext}})$ is only unique up to isomorphism. We "canonify" it (following G. Rousseau) by viewing (which we will adopt from now on) $\mathbb{A}_G$ as $V_G$ with a marked origin $\{0\}$.*

**Definition** II.3.7.1. *The extended building is the following product of a poly-simplicial complex and a real vector space*

$$\mathcal{B}(\mathbf{G},F)_{\text{ext}} = \mathcal{B}(\mathbf{G},F)_{\text{red}} \times V_G.$$

*The group $\mathbf{G}(F)$ acts isometrically on it as follows:*

$$g\cdot(x,v) = (g\cdot x, v+\nu_G(g)) \quad \forall g \in \mathbf{G}(F), x \in \mathcal{B}(\mathbf{G},F)_{\text{red}}, v \in V_G.$$

A facet (resp. alcove, Weyl chamber, apartment) of $\mathcal{B}(\mathbf{G},F)_{\text{ext}}$ is defined to be a product of a same object in $\mathcal{B}(\mathbf{G},F)_{\text{red}}$ and $V_G$. For instance let us fix the extended apartment $\mathcal{A}_{\text{ext}} = \mathcal{A} \times V_G$. We identify $\mathcal{B}(\mathbf{G},F)_{\text{red}}$ with the subset $\mathcal{B}(\mathbf{G},F)_{\text{red}} \times \{0\}$ in $\mathcal{B}(\mathbf{G},F)_{\text{ext}}$. Then, the stabilizer of $\mathcal{B}(\mathbf{G},F)_{\text{red}}$ for the action of $\mathbf{G}(F)$ in the extended building is exactly $\ker\nu_G$. In this building the minimal dimensional facets are of the form $\mathcal{F}_x := \{x\} \times V_G$ for some vertex $x \in \mathcal{B}(\mathbf{G},F)_{\text{red}}$. The stabilizer of $(x,0)$ is equal to $P_x \cap \ker\nu_G$, where $P_x$ is the subgroup defined in §II.3.3.

## II.3.8 The extended affine Weyl group

We define the extended affine Weyl group of $\mathbf{S}$ to be $\nu_{N,\text{ext}}(\mathbf{N}(F)) \subset \text{Aff}(\mathbb{A}_{\text{ext}}(\mathbf{G},\mathbf{S}))$, hence

$$\widetilde{W}_{\text{aff}} \simeq \mathbf{N}(F)/\ker\nu_{N,\text{ext}} = \mathbf{N}(F)/(\ker\nu_N \cap \mathbf{G}(F)^1) = \mathbf{N}(F)/\mathbf{M}(F)^1.$$

There are two important decompositions for $\widetilde{W}_{\text{aff}}$. If $a \in \mathbb{A}_{\text{ext}}(\mathbf{G}, \mathbf{S})$ is special then using the exact sequence (§II.3.2)

$$0 \to \mathbf{M}(F)/\,\mathbf{M}(F)^1 \to \widetilde{W}_{\text{aff}} \to W \to 0,$$

one obtains the following isomorphism

$$\widetilde{W}_{\text{aff}} \simeq \underline{\Lambda}_M \rtimes W^a_{\text{aff}},$$

where, $\underline{\Lambda}_M = \mathbf{M}(F)/\,\mathbf{M}(F)^1 \simeq \nu_{N,\text{ext}}(\mathbf{M}(F))$, with this decomposition one can view $\widetilde{W}_{\text{aff}}$ as a group of affine-linear transformations on $\mathbb{A}_{\text{ext}}(\mathbf{G}, \mathbf{S})$ marked with $a$ as the "origin".

Consider the subgroup $\Omega_{\mathfrak{a}} \subset \widetilde{W}_{\text{aff}}$ that fixes $\mathfrak{a} \times \{0\} \in \mathbb{A}_{\text{ext}}(\mathbf{G}, \mathbf{S})$. Taking Fr-fixed points of the external semidirect product given in [HR08, Lemma 14] (relative to the alcove $\mathfrak{a}$) yields the following external semidirect product

$$\widetilde{W}_{\text{aff}} \simeq W_{\text{aff}} \rtimes \Omega_{\mathfrak{a}}.$$

## II.3.9 Parahoric subgroups

### II.3.9.1 Further notations

Let $F^{un}$ denote the maximal unramified extension of $F$ contained in $F^{sep}$, and $L$ the completion of $F^{un}$ with respect to the valuation on $F^{un}$ which extends the normalized valuation on $F$. The residue field of $F$ is perfect, thus $F^{un} = F^{\text{sh}}$ is the strict henselization of $F$ in the fixed separable closure $F^{sep}$. Let $L^{sep}$ a separable closure of $L$ containing $F^{sep}$. The arithmetic Frobenius automorphism $\sigma \in \text{Gal}(F^{un}/F)$[30] extends continuously to an automorphism of $L$ over $F$, also denoted $\sigma$. Write $\text{In} = \text{Gal}(F^{sep}/F^{un})$ for the inertia subgroup of $\text{Gal}(F^{sep}/F)$. Since $L^{sep} = L \otimes_{F^{un}} F^{sep}$, one can identify the inertia subgroup In with $\text{Gal}(L^{sep}/L)$.

### II.3.9.2 The Kottwitz homomorphism

Let $\mathbf{H}$ be any connected reductive $F$-group, $\widehat{\mathbf{H}}$ be its connected Langlands dual (§II.1.12) and $Z(\widehat{\mathbf{H}})$ be the center of $\widehat{\mathbf{H}}$.

[30] Which induces $x \mapsto x^q$ on the residue field of $F^{un}$.

Kottwitz defines in [Kot97, §7.7] a functorial surjective homomorphism

$$\kappa_{\mathbf{H}} : \mathbf{H}(L) \to X^*(Z(\widehat{\mathbf{H}})^{\mathrm{In}}) = X^*(Z(\widehat{\mathbf{H}}))_{\mathrm{In}},$$

where, the subscript In indicates coinvariants of the $\mathrm{Gal}(F^{sep}/F)$-module $X^*(Z(\widehat{\mathbf{H}}))$, this latter is the Borovoi fundamental algebraic group (usually denoted $\pi_1(\mathbf{H})$) and it is isomorphic to the quotient $X_*(\mathbf{T})$ for a maximal $F$-torus $\mathbf{T}$ by the coroots lattice. The group $\pi_1(\mathbf{H})$ acquires an action of $\mathrm{Gal}(F^{sep}/F)$ via its representation as $X_*(\mathbf{T})/\sum_{\alpha\in\Phi(\mathbf{H}_{F^{sep}},\mathbf{T})}\mathbb{Z}\alpha^\vee$. There is a canonical surjective homomorphism

$$q_{\mathbf{H}} \colon X^*(Z(\widehat{\mathbf{H}}))_{\mathrm{In}} \to \mathrm{Hom}_{\mathbb{Z}}(X^*(\mathbf{H})^{\mathrm{In}}, \mathbb{Z})$$

whose kernel is the torsion subgroup of $X^*(Z(\widehat{\mathbf{H}}))_{\mathrm{In}}$ [Kot97, 7.4.4][31]. Kottwitz shows in [Kot97, §7.4] that the above two homomorphisms sit in the following commutative diagram:

$$\begin{tikzcd}
\mathbf{H}(L) \arrow[rr, two heads, "\kappa_{\mathbf{H}}"] \arrow[drr, "-\nu_{\mathbf{H}}"'] & & X^*(Z(\widehat{\mathbf{H}}))_{\mathrm{In}} \arrow[d, two heads, "q_{\mathbf{H}}"] \\
& & \mathrm{Hom}_{\mathbb{Z}}(X^*(\mathbf{H})^{\mathrm{In}}, \mathbb{Z})
\end{tikzcd}$$

where, $\nu_{\mathbf{H}} \colon \mathbf{H}(F) \to \mathrm{Hom}_{\mathbb{Z}}(X^*(\mathbf{H})^{\mathrm{In}}, \mathbb{Z})$ is the natural homomorphism characterized by

$$\langle \nu_{\mathbf{H}}(h), \chi\rangle = \nu_{\mathbf{H}}(h)(\chi) = -\omega(\chi(h)), \text{ for all } h \in \mathbf{H}(L) \text{ and } \chi \in X^*(\mathbf{H})^{\mathrm{In}}.$$

Therefore, $\ker \kappa_{\mathbf{H}} \subset \ker \nu_{\mathbf{H}}$, and

$$\left(X^*(Z(\widehat{\mathbf{H}}))_{\mathrm{In}}\right)_{\mathrm{tors}} = \ker \nu_{\mathbf{H}} / \ker \kappa_{\mathbf{H}}.$$

**Remark** II.3.9.1. *Note here that our $\nu_{\mathbf{H}}$ differs in a sign from the map $v_H$ in [Kot97, 7.4.3].*

Since our conneced reductive group $\mathbf{H}$ is defined over $F$, the restriction of $\kappa_{\mathbf{H}}$ to $\mathbf{H}(F)$ provides a surjective homomorphism $\kappa_H$ that sits in the following commutative diagram

[31] The codomain of (7.4.4) in *loc. cit.* is actually $\mathrm{Hom}_{\mathbb{Z}}(X_*(Z(\widehat{\mathbf{H}}))^{\mathrm{In}}, \mathbb{Z}))$, so one needs to use the isomorphism $X_*(Z(\widehat{\mathbf{H}})) \simeq X^*(\mathbf{H})$.

(see [Kot97, §7.7] for the surjectivity of $\kappa_H$, see also [Ros15, §2.7])

$$\begin{tikzcd} \mathbf{H}(F) \arrow[r, two heads, "\kappa_H"] \arrow[dr, "-\nu_H"'] & (X^*(Z(\widehat{\mathbf{H}}))_{\mathrm{In}})^{\langle\sigma\rangle} \arrow[d, "q_H"] \\ & \mathrm{Hom}_{\mathbb{Z}}(X^*(\mathbf{H})_F, \mathbb{Z}) \end{tikzcd}$$

We denote by $\mathbf{H}(L)_1$ (respectively $\mathbf{H}(L)^1$, $\mathbf{H}(F)_1 = \mathbf{H}(F) \cap \ker \kappa_{\mathbf{H}}$ and $\mathbf{H}(F)^1$) the kernel of $\kappa_{\mathbf{H}}$ (respectively $\nu_{\mathbf{H}}$, $\kappa_H$ and $\nu_H$). Therefore, [HV15, §3.2, Lemma],

$$\mathbf{H}(F)^1 = \mathbf{H}(L)^1 \cap \mathbf{H}(F) = \left\{ h \in \mathbf{H}(F) \colon \kappa_H(h) \in \left( X^*(Z(\widehat{\mathbf{H}}))^{\sigma}_{\mathrm{In}} \right)_{\mathrm{tors}} \right\}.$$

Set $\Lambda_H := \mathbf{H}(F)/\mathbf{H}(F)_1$, thus

$$\mathbf{H}(F)^1/\mathbf{H}(F)_1 = (\Lambda_H)_{\mathrm{tors}} \simeq \left( X^*(Z(\widehat{\mathbf{H}}))^{\sigma}_{\mathrm{In}} \right)_{\mathrm{tors}}.$$

### II.3.9.3 Descente and passage to completion

Consider the extended building $\mathcal{B}(\mathbf{G}, F^{un})_{\mathrm{ext}}$ of the group $\mathbf{G}_{F^{un}} = \mathbf{G} \times_F F^{un}$. This building is equipped with an action of $\mathbf{G}(F^{un}) \rtimes \langle\sigma\rangle$. Moreover, there is a natural $\mathbf{G}(F)$-equivariant embedding $\iota \colon \mathcal{B}(\mathbf{G}, F)_{\mathrm{ext}} \hookrightarrow \mathcal{B}(\mathbf{G}, F^{un})_{\mathrm{ext}}$ such that [BT84, 5.1.25]

$$\iota(\mathcal{B}(\mathbf{G}, F)_{\mathrm{ext}}) = \mathcal{B}(\mathbf{G}, F^{un})^{\sigma}_{\mathrm{ext}}.$$

In other words, the extended building of $\mathbf{G}$ over $F$ is identified with the fixed points of $\sigma \in \mathrm{Gal}(F^{un}/F)$ (in particular of $\sigma$) in the building of $\mathbf{G}$ over $F^{un}$.

Since $F^{un}$ is henselian, Rousseau has shown in [Rou77, proposition 2.3.9] that $\mathbf{G}$ has the same relative rank over $F^{un}$ and over $L$, and hence by [Rou77, proposition 2.3.9] the building $\mathcal{B}(\mathbf{G}, F^{un})_{\mathrm{ext}}$ identifies canonically with $\mathcal{B}(\mathbf{G}, L)_{\mathrm{ext}}$ as $\mathbf{G}(F^{un})$-space. Using this identification, we see that the extended building of $\mathbf{G}$ over $F$, can also be identified with the fixed points of $\sigma \in \mathrm{Aut}(L/F)$ in the building of $\mathbf{G}$ over $L$, i.e.

$$\iota(\mathcal{B}(\mathbf{G}, F)_{\mathrm{ext}}) = \mathcal{B}(\mathbf{G}, L)^{\sigma}_{\mathrm{ext}}.$$

This gives a bijection between the set of $\sigma$-stable facets in $\mathcal{B}(\mathbf{G}, L)_{\mathrm{ext}}$ and the set of facets in $\mathcal{B}(\mathbf{G}, F)_{\mathrm{ext}}$. For more details one can consult [Pra17].

### II.3.9.4 $F$-Parahoric subgroups

In [BT84, 4.6.28, 5.2.6], Bruhat and Tits associated two subgroups to any non-empty $\sigma$-stable bounded set $\Omega$ which is contained in an appartement of $\mathcal{B}(\mathbf{G}, F^{un})_{\text{ext}}$:

1. The fixator subgroup
$$P(F^{un})_{\Omega} := \{g \in \mathbf{G}(F^{un}) \colon g \cdot a = a, \forall a \in \Omega\} \subset \mathbf{G}(F^{un}).$$
They also showed in *loc. cit.*, the existence of a smooth affine group scheme $\mathbb{P}_{\Omega}$ over $\operatorname{Spec} \mathcal{O}_F$ with generic fiber $\mathbf{G}$, and which is uniquely characterized by the property
$$\mathbb{P}_{\Omega}(\mathcal{O}_{F^{un}}) = P(F^{un})_{\Omega}.$$
2. The parahoric subgroup ("connected fixator")
$$P(F^{un})^{\circ}_{\Omega} := \mathbb{P}^{\circ}_{\Omega}(\mathcal{O}_{F^{un}}) \subset P(F^{un})_{\Omega},$$
where $\mathbb{P}^{\circ}_{\Omega}$ is the identity component of $\mathbb{P}_{\Omega}$.

A $F$-parahoric subgroup of $\mathbf{G}$ is by definition [BT84, under 5.2.6] the "connected fixator" of a facet $\mathcal{F} \subset \iota(\mathcal{B}(\mathbf{G}, F)_{\text{ext}})$.

**Lemma** II.3.9.1. *Two facets $\mathcal{F}, \mathcal{F}' \subset \iota(\mathcal{B}(\mathbf{G}, F)_{\text{ext}})$ are equal if and only if*
$$P(F^{un})^{\circ}_{\mathcal{F}} \cap \mathbf{G}(F) = P(F^{un})^{\circ}_{\mathcal{F}'} \cap \mathbf{G}(F).$$

*Proof.* See [BT84, 5.2.8]. □

This lemma justifies (by misuse of language) the following definition:

**Definition** II.3.9.1. *A parahoric subgroup of $\mathbf{G}(F)$ is the intersection*
$$K_{\mathcal{F}} := P(F^{un})^{\circ}_{\mathcal{F}} \cap \mathbf{G}(F),$$
*for some facet $\mathcal{F} \subset \iota(\mathcal{B}(\mathbf{G}, F)_{\text{ext}})$. When the facet $\mathcal{F}$ is an alcove, the parahoric subgroup $K_{\mathcal{F}}$ is called an Iwahori subgroup and will be denoted $I_{\mathcal{F}}$ instead.*

**Proposition** II.3.9.1. *We have $P(F^{un})^{\circ}_{\mathcal{F}} = P(F^{un})_{\mathcal{F}} \cap \mathbf{G}(F^{un})_1$.*

*Proof.* See [HR08, Proposition 3, remarks 3 and 11]. □

**Remark** II.3.9.2. *As pointed out in [HR08, Remark 4], Proposition II.3.9.1 remains valid if we replace $\mathcal{F}$ by any bounded subset $\Omega \subset\subset \iota(\mathcal{B}(\mathbf{G}, F)_{\mathrm{ext}})$ which is contained in an apartment.*

This proposition induces another characterization of parahoric subgroups of $\mathbf{G}(F)$:

**Corollary** II.3.9.1. *Parahoric subgroups of $\mathbf{G}(F)$ are fixators of facets in the kernel of the Kottwitz homomorphism, i.e. for any facet $\mathcal{F} \subset \mathcal{B}(\mathbf{G}, F)_{\mathrm{ext}}$ we have $K_{\mathcal{F}} = \mathbb{P}_{\mathcal{F}}(F) \cap \mathbf{G}(F)_1$.*

*Proof.* The corollary follows from the following equalities:

$$\begin{aligned} K_{\mathcal{F}} &= P(F^{un})^{\circ}_{\mathcal{F}} \cap \mathbf{G}(F) \\ &= P(F^{un})_{\mathcal{F}} \cap \mathbf{G}(F^{un})_1 \cap \mathbf{G}(F) \quad \text{(Proposition II.3.9.1)} \\ &= P(F^{un})_{\mathcal{F}} \cap \mathbf{G}(F)_1 \\ &= \mathbb{P}_{\mathcal{F}}(F^{un}) \cap \mathbf{G}(F)_1 \\ &= \mathbb{P}_{\mathcal{F}}(F) \cap \mathbf{G}(F)_1. \end{aligned}$$ □

**Remark** II.3.9.3. *All parahoric subgroups of $\mathbf{G}(F)$ generates $\mathbf{G}(F)_1$ [HR08, Lemma 17].*

The map that associates to a facet $\mathcal{F}$ its parahoric subgroup $K_{\mathcal{F}}$ is decreasing, in particular, if $\mathcal{F}$ is any facet lying in the closure[32] of the alcove $\mathfrak{a} \times V_G \subset \mathcal{B}(\mathbf{G}, F)_{\mathrm{ext}}$ then we have $I_{\mathfrak{a}\times V_G} \subset K_{\mathcal{F}}$. In addition, the action of an element $g \in \mathbf{G}(F)$ on a facet $\mathcal{F}$ translates for parahorics to $K_{g\cdot\mathcal{F}} = gK_{\mathcal{F}}g^{-1}$.

**Lemma** II.3.9.2. *The subgroup $\mathbf{M}(F)_1 := \ker \kappa_M$ is the unique parahoric of $\mathbf{M}(F)$, and it is a finite index subgroup of $\mathbf{M}(F)^1 = \ker \nu_M$ (The maximal compact subgroup of $\mathbf{M}(F)$). In addition, for any facet $\mathcal{F}$ in $\mathcal{A}_{\mathrm{ext}}$ the apartment corresponding to $\mathbf{S}$, we have*

$$\mathbf{M}(F) \cap K_{\mathcal{F}} = \mathbf{M}(F)_1.$$

*Proof.* See [HR10, Lemmas 4.1.1, 4.2.1]. □

By [BT84, §5.2.4], we get another characterization of parahoric subgroups:

**Proposition** II.3.9.2. *The parahoric subgroup of $\mathbf{G}(F)$ associated to a facet $\mathcal{F} \subset \mathcal{A}_{\mathrm{ext}}$ is equal to the subgroup generated by $\mathbf{M}(F)_1$ and $U_{\mathcal{F}}$, more precisely it has the following*

[32] The closure of an alcove consists of points lying in it and in all its faces.

*decomposition*

$$K_{\mathcal{F}} = U_{\mathcal{F}}\,\mathbf{M}(F)_1 = U^+_{\mathcal{F}}U^-_{\mathcal{F}}U^+_{\mathcal{F}}\,\mathbf{M}(F)_1 = U^+_{\mathcal{F}}U^-_{\mathcal{F}}(\mathbf{N}(F)\cap K_{\mathcal{F}}).$$

*such that the factors commute in the right factorization and the product maps*

$$\textstyle\prod_{\alpha\in\Phi_{\mathrm{red}}\cap\Phi^{\pm}} U_{\alpha+f_{\mathcal{F}}(\alpha)} \xrightarrow{\ \sim\ } U^{\pm}_{\mathcal{F}} = U_{\mathcal{F}}\cap\mathbf{U}(F)^{\pm} = K_{\mathcal{F}}\cap\mathbf{U}(F)^{\pm}.$$

*homeomorphism whatever ordering of the factors we take.*

Remark II.3.9.4. *As observed by Haines [Hai09, §6], the reference [BT84, §5.2.4] contains a typographical error. All of the "hats" in the four displayed equations in* loc. cit. *should be removed.*

Remark II.3.9.5. *Let $\Omega\subset\mathcal{A}_{\mathrm{ext}}$ be any bounded subset containing a facet $\mathcal{F}$, and set $K_\Omega$ for its $\mathbf{G}(F)_1$-fixator. By Remark II.3.9.2 it is precisely the $\mathcal{O}_F$-points of the "connected fixator" $\mathbb{P}^\circ_\Omega$ introduced in §II.3.9.4, as such we also have $K_\Omega = P(F^{un})^\circ_\Omega\cap\mathbf{G}(F)_1$. Applying [BT84, §5.2.4] we have*

$$K_\Omega = U_\Omega\,\mathbf{M}(F)_1 = U^+_\Omega U^-_\Omega U^+_\Omega\,\mathbf{M}(F)_1 = U^+_\Omega U^-_\Omega(\mathbf{N}(F)\cap K_\Omega).$$

*Accordingly,*

$$\mathbf{M}(F)_1\subset K_\Omega\cap\mathbf{M}(F)\subset K_{\mathcal{F}}\cap\mathbf{M}(F) \overset{Lem. II.3.9.2}{=} \mathbf{M}(F)_1.$$

*Therefore, $\mathbf{M}(F)_1 = K_\Omega\cap\mathbf{M}(F)$.*

Since there is a one-to-one correspondence between systems of positive roots for $\Phi(\mathbf{G},\mathbf{S})$ and vectorial chambers in $\mathbb{A}_{\mathrm{ext}}(\mathbf{G},\mathbf{S})$ with apex $a_\circ$ (see Remark II.2.6.1), we may and will refine our choice of $\mathfrak{a}$ to be the unique alcove with apex $a_\circ$ such that the set

$$\{\alpha\in\Phi\colon \alpha \text{ is } \mathfrak{a}\text{-positive}\},$$

equals to the fixed system of positive roots $\Phi^+$.

Lemma II.3.9.3. *We have*

$$\mathbf{N}(F)\cap I_{\mathfrak{a}\times V_G} = \mathbf{M}(F)_1.$$

*Proof.* Let $n\in\mathbf{N}(F)\cap I_{\mathfrak{a}\times V_G}\subset\mathbf{N}(F)\cap K_{a_\circ\times V_G}$. But $a_\circ$ being special implies that the canonical injection

$$\mathbf{N}(F)\cap K_{a_\circ\times V_G}/\,\mathbf{M}(F)\cap K_{a_\circ\times V_G}\hookrightarrow W$$

is an isomorphism [BT72, §4.4.2]. But Lemma II.3.9.2 asserts that $\mathbf{M}(F)\cap K_{a_\circ\times V_G} = \mathbf{M}(F)_1$. Fixing $\mathfrak{a}$ in the reduced appartement $\mathcal{A}_{\mathrm{red}}$ shows that the vectorial part of $\nu_N(n)$

is trivial, thus $n$ acts on $\mathcal{A}_{\text{red}}$ as a rotation:

$$\nu_N(n)(a) = w(n)(a - a_\circ) + a_\circ, \quad \forall a \in \mathcal{A}_{\text{red}}.$$

By assumption $\nu_N(n)(a) = a$ for any $a \in \bar{\mathfrak{a}}$, which means that $w(n)(a - a_\circ) = a - a_\circ$ for all $a \in \bar{\mathfrak{a}}$, in particular for all $a \in \operatorname{vert}(\mathfrak{a}) = \{\text{vertices of } \mathfrak{a}\}$. But the set

$$\{a - a_\circ : a \in \operatorname{vert}(\mathfrak{a}) \setminus \{a_\circ\}\}$$

forms a basis for $V$ and hence $w(n)$ is the identity on $V$. Accordingly,

$$\mathbf{N}(F) \cap I_{\mathfrak{a}\times V_G} \subset \mathbf{M}(F)_1 \overset{\text{Lem. II.3.9.2}}{=} \mathbf{M}(F) \cap I_{\mathfrak{a}\times V_G} \subset \mathbf{N}(F) \cap I_{\mathfrak{a}\times V_G}. \quad \square$$

**Corollary** II.3.9.2 (Iwahori factorizations). *The Iwahori subgroup $I_{\mathfrak{a}\times V_G}$ admits the following decomposition*

$$I_{\mathfrak{a}\times V_G} = U^+_{\mathfrak{a}\times V_G}\mathbf{M}(F)_1 U^-_{\mathfrak{a}\times V_G} = U^-_{\mathfrak{a}\times V_G}\mathbf{M}(F)_1 U^+_{\mathfrak{a}\times V_G},$$

*such that the factors commute and the product maps*

$$\prod_{\alpha\in\Phi^+\cap\Phi_{\text{red}}} U_{\alpha+0} \to U^+_{\mathfrak{a}\times V_G} = I_{\mathfrak{a}\times V_G} \cap \mathbf{U}(F)^+$$

*and*

$$\prod_{\alpha\in\Phi^-\cap\Phi_{\text{red}}} U_{\alpha+n_\alpha^{-1}} \to U^-_{\mathfrak{a}\times V_G} = I_{\mathfrak{a}\times V_G} \cap \mathbf{U}(F)^-,$$

*are homeomorphisms.*

*We also have a quiet similar decomposition for the special maximal parahoric subgroup associated to the minimal facet $\{a_\circ\} \times V_G$:*

$$\begin{aligned} K_{\{a_\circ\}\times V_G} &= U^+_{\{a_\circ\}\times V_G} U^-_{\{a_\circ\}\times V_G} U^+_{\{a_\circ\}\times V_G}\mathbf{M}(F)_1 \\ &= U^-_{\{a_\circ\}\times V_G} U^+_{\{a_\circ\}\times V_G} U^-_{\{a_\circ\}\times V_G}\mathbf{M}(F)_1, \end{aligned}$$

*and the product maps:*

$$\prod_{\alpha\in\Phi^\pm\cap\Phi_{\text{red}}} U_{\alpha+0} \to U^\pm_{\{a_\circ\}\times V_G} = K_{\{a_\circ\}\times V_G} \cap \mathbf{U}(F)^\pm.$$

*are homeomorphisms.*

Recall that $n_\alpha$ is the positive integer defined in Proposition II.3.3.2 and $U_{\alpha+r} = \{u \in \mathbf{U}_\alpha(F) : \varphi_\alpha(u) \geq r\}$ for $\alpha \in \Phi, r \in \mathbb{R}$.

*Proof.* The first equalities of the two Iwahori factorization, namely

$$K_{\{a_\circ\}\times V_G} = U^+_{\{a_\circ\}\times V_G} U^-_{\{a_\circ\}\times V_G} U^+_{\{a_\circ\}\times V_G}\mathbf{M}(F)_1 \text{ and } I_{\mathfrak{a}\times V_G} = U^+_{\mathfrak{a}\times V_G}\mathbf{M}(F)_1 U^-_{\mathfrak{a}\times V_G},$$

follows from Proposition II.3.9.2 and Lemma II.3.9.3. Using the opposite root system $\Phi^-$ instead of $\Phi^+$, one gets the other two equalities. Finally, to prove that the product maps are homeomorphisms, we need (4) of Proposition II.3.4.1 together with Example II.3.4.1 for the values of $f_{\mathfrak{a}}(\alpha)$ for all $\alpha \in \Phi_{\mathrm{red}}$. $\square$

# CHAPTER III

# THE RING OF $\mathbb{U}$-OPERATORS AND HECKE ALGEBRAS

In this chapter, we define the ring of $\mathbb{U}$-operators and prove their integrality over the *spherical* Hecke algebra.

The central tool we use for defining and studying these operators, is the universal unramified principal series right $\mathcal{H}_I(\mathbb{Z})$-module $\mathcal{M}_I(\mathbb{Z})$ (§III.8). We will be studying $\mathcal{M}_I(\mathbb{Z})$ following the exposition of [HKP10], which treats the case of a split reductive group. Therefore, we will have to first reformulate and then establish these properties in our situation where the group is no longer required to split over $F$. We end the chapter by relating the $\mathbb{U}$-operators to the Hecke polynomial defined by Blasius and Rogawski [BR94, §6], a polynomial that was introduced in order to generalize the classical Eichler–Shimura relation for modular curves by conjecturing that it annihilates the Frobenius correspondence acting on the $\ell$-adic étale cohomology. We will show that this Hecke polynomial has also a root in $\mathbb{U}$.

## Notations

In addition to the notation adopted in Chapter II.3, we also introduce the following, that we hope will lighten a bit the exposition: For any algebraic $F$-groups $\mathbf{H}$ (bold style), $H = \mathbf{H}(F)$ will denote its $F$-points. Let $K$ be the special maximal parahoric subgroup $K_{\{a_\circ \times V_G\}}$ (§II.3.9.4) associated to the minimal dimensional facet $a_\circ \times V_G \subset \mathbb{A}_{\text{ext}}(\mathbf{G}, \mathbf{S})$, where $a_\circ$ is the special vertex introduced below corollary II.3.3.1 in §II.3.5. Let $I = I_{\mathfrak{a} \times V_G}$ be the Iwahori subgroup corresponding to the alcove $\mathfrak{a} \times V_G \subset \mathbb{A}_{\text{ext}}(\mathbf{G}, \mathbf{S})$ fixed before corollary II.3.9.2.

## III.1 A Tits system

Define[1]

$$\mathcal{S}(\mathfrak{a}) = \{s_\alpha \colon \alpha \in \Phi_{\text{aff}} \text{ and } H_\alpha \text{ is a wall of } \mathfrak{a}\}.$$

By [Vig16, §3.9] the action of $N_1 := N \cap G_1$ on $\mathbb{A}_{\text{red}}(\mathbf{G}, \mathbf{S})$ identifies $W_{\text{aff}}$ with $N_1/M_1$ and $\mathcal{S}(\mathfrak{a}) \subset W_{\text{aff}}$ with a subset $\mathcal{S} \subset N_1/M_1$.

A good part of the following two sections relies heavily on the following theorem of Bruhat

[1] Recall that a wall of $\mathfrak{a}$ is a hyperplane $H$ containing a face of $\mathfrak{a}$ (§II.3.5).

and Tits [BT72, Proposition 5.2.12 (i)]:

**Theorem** III.1.0.1. *The quadruple $(G_1, I, N_1, \mathcal{S})$ is a Tits system (see Definition II.2.7.1).*

In particular, the pair $(N_1/M_1, \mathcal{S})$ (or equivalently $(W_{\text{aff}}, \mathcal{S}(\mathfrak{a}))$) is a Coxeter system [Bou68, §3], in addition to the following properties:

(T1) The subgroup $G_1$ is generated by $I$ and $N_1$,

(T2) the elements of $\mathcal{S}$ have order 2 and generate $N_1/M_1$,

(T3) For all $s \in \mathcal{S}, w \in N_1/M_1$, we have $sIw \subset IwI \sqcup IswI$,

(T4) For all $s \in \mathcal{S}$, we have $sIs \not\subseteq I$.

## III.2 Iwahori–Weyl group

Define the Iwahori–Weyl group for $G$ as

$$\widetilde{W} := N/M_1.$$

Let us recall the Bruhat decompositions for $G$:

**Proposition** III.2.0.1 (Bruhat decompositions for $G$). *Let $B = MU^+$ be the minimal parabolic subgroup of $G$ with Levi factor $M$, and unipotent radical $U^+$. We have*

$$G = BNB = INI.$$

*Moreover, the maps $n \mapsto BnB$ and $n \mapsto InI$ induces the following bijections*

$$W \simeq B\backslash G/B \text{ and } \widetilde{W} \simeq I\backslash G/I.$$

*Proof.* We refer to [HR08, Remark 9] and [Vig16, Proposition 3.35]. □

We now present two semidirect product decompositions of the Iwahori–Weyl group similar to the one for the extended Weyl group in §II.3.8. These decompositions will be useful for expression presentations of the Iwahori–Hecke algebra. Here is the first one:

**Lemma** III.2.0.1. *The Weyl-Iwahori subgroup has a natural structure of a quasi-Coxeter group*[2]

$$\widetilde{W} \simeq W_{\text{aff}} \rtimes \widetilde{\Omega} \simeq W_{\text{aff}} \rtimes \Lambda_G,$$

[2] that is a semi-direct product of a Coxeter group with an abelian group.

*where, $W_{\mathrm{aff}} \simeq N_1/M_1$ denotes the affine Weyl group of $\Phi_{\mathrm{aff}}$, $\widetilde{\Omega} \subset \widetilde{W}$ is the subset consisting of stabilizers of the alcove $\mathfrak{a}$ and $\Lambda_G = G/G_1 (\simeq X^*(Z(\widehat{\mathbf{G}}))^{\sigma}_{In}$ see §II.3.9.2).*

*Proof.* The previous decomposition can be deduced from [HR08, Lemma 14] by taking the $\sigma$-fixed points. For a more comprehensive proof we refer to [Vig16, §3.9], where $\Lambda_G$ must be identified with $\widetilde{\Omega} = N/N_1$ (see Proposition3.36 in *loc. cit*, where $\widetilde{\Omega}$ is denoted by $\Omega$). □

**Remark** III.2.0.1. *Since $K$ is a special subgroup of $G$, the canonical injection*

$$N \cap K/M \cap K \hookrightarrow W$$

*is an isomorphism [BT72, §4.4.2]. Therefore, from now on, we may and will assume that every representative in $N$ of an element $w \in W = N/M$ lies in $K$, such a representative is determined up to multiplication by $M_1 = M \cap K$ (Lemma II.3.9.2).*

**Lemma** III.2.0.2 (Bernstein). *The Iwahori–Weyl group $\widetilde{W}$ admits the following decomposition*

$$\widetilde{W} = \Lambda_M \rtimes W,$$

*where, as in the notation introduced in §II.3.9.2, $\Lambda_M = M/M_1$.*

*Proof.* We have an exact short sequence

$$1 \longrightarrow M/M_1 \longrightarrow N/M_1 \xrightarrow{\pi_W} N/M \longrightarrow 1\ .$$

Using the canonical isomorphism $W \simeq N \cap K/M \cap K$ (Remark III.2.0.1), one gets a homomorphism $\varpi^W \colon W \to N/M_1$ , such that $\pi_W \circ \varpi^W = \mathrm{Id}_W$. Hence, the above short exact sequence splits:

$$\widetilde{W} = M/M_1 \rtimes N \cap K/M_1 \simeq \Lambda_M \rtimes W. \quad \square$$

## III.3 Double cosets

Denote by $\ell \colon W_{\mathrm{aff}} \simeq N_1/M_1 \to \mathbb{N}$ the length function of the Coxeter system $(W_{\mathrm{aff}}, \mathcal{S}(\mathfrak{a}))$[3](Recall that $(W_{\mathrm{aff}}, \mathcal{S}(\mathfrak{a}))$ and $(N_1/M_1, \mathcal{S})$ are isomorphic Coxeter systems). Inflate the map $\ell$ to a

[3]Recall that if $(W, S)$ is a Coxeter system, a reduced word for an element $w \in W$ is a minimal length expression of $w$ as a product of elements of $S$, the length $\ell(w)$ is the length of a reduced word.

length function $\widetilde{W} \simeq W_{\text{aff}} \rtimes \widetilde{\Omega} \to \mathbb{N}$, for which $\widetilde{\Omega} \subset \widetilde{W}$ is exactly the subset of elements of length equal to 0[4]:

$$\ell(w) = \ell(w_{\text{aff}}) \text{ if } w = w_{\text{aff}}u \text{ with } w_{\text{aff}} \in W_{\text{aff}}, u \in \widetilde{\Omega}.$$

Furthermore, we consider the Chevalley–Bruhat (partial) order[5] on $W_{\text{aff}} \simeq N_1/M_1$. Now extend the Chevalley–Bruhat order to the Iwahori–Weyl group $\widetilde{W}$ as follows: we say $(w_1, \lambda_1) \leq (w_2, \lambda_2) \in \widetilde{W} \simeq W_{\text{aff}} \rtimes \Lambda_G$ if and only if $w_1 \leq w_2$ in $W_{\text{aff}}$ and $\lambda_1 = \lambda_2$ in $\Lambda_G$.

**Lemma** III.3.0.1. *For every $w \in \widetilde{W}$ and $s \in \mathcal{S}$ we have*[6]

$$IsIwI = \begin{cases} IswI, & \textit{if } w < sw, \\ IwI \sqcup IswI, & \textit{if } sw < w. \end{cases}$$

*Proof.* We will use [Bou68, Ch. IV §2, Exercice 8, page 48]. Recall that $G_1$ (being the kernel of $\kappa_G$) is normal in $G$. Let us verify that for every $g \in G$, there exists $h \in G_1$ such that $hIh^{-1} = gIg^{-1}$ and $hN_1h^{-1} = gN_1g^{-1}$. Let $N_{g\cdot\mathfrak{a}}$ be the $G$-stabilizer of the alcove $g \cdot \mathfrak{a}$, this is also the $G$-normalizer of $I$. By [Vig16, Proposition 3.36 (2)], we have $G = G_1N_{g\cdot\mathfrak{a}}$, hence $g = hn$ for some $n \in N_{g\cdot\mathfrak{a}}$ and $h \in G_1$ and $g \cdot \mathfrak{a} = h \cdot \mathfrak{a}$ i.e. $gIg^{-1} = hIh^{-1}$. By *loc. cit.* $N_{g\cdot\mathfrak{a}}$ normalizes $N_1$ and so $gN_1g^{-1} = hnN_1n^{-1}h^{-1} = hN_1h^{-1}$. Recall that by Theorem III.1.0.1, the quadruplet $(G_1, I, N_1, \mathcal{S})$ is a Tits system, therefore by [Bou68, Ch. IV §2 Exercice 8, (c) and (d)] (taking $\widetilde{G}$ for $G$ and $G$ for $G_1$) we deduce that

$$IsIwI \subset IwI \sqcup IswI,$$

for $s \in \mathcal{S}$ and $w \in \widetilde{W}$. From this one can deduce (see [Bou68, Ch. VI §2, Exercice 11 (d), page 49]):

$$IsIwI = \begin{cases} IswI, & \text{if } w < sw \quad (\ell(sw) = \ell(w) + 1), \\ IwI \sqcup IswI, & \text{if } sw < w \quad (\ell(sw) = \ell(w) - 1). \end{cases} \qquad \square$$

**Remark** III.3.0.1. *The first equality in the lemma above can be easily generalized: for any $w, w' \in \widetilde{W}$*

$$IwIw'I = Iww'I, \quad \textit{if} \quad \ell(ww') = \ell(w) + \ell(w'). \quad \square$$

---

[4]Equivalently, the length of an element in $w \in \widetilde{W}$ is the number of walls between the fixed alcove $\mathfrak{a}$ and the alcove $w(\mathfrak{a})$.

[5]Let $(W, S)$ be a Coxeter system and define a partial order on $W$ as follows: Fix a reduced word $w = s_1s_2 \ldots s_k$. We say $v \leq w$ if and only if there is a reduced subword $s_{i_1}s_{i_2}\ldots s_{i_j} = v$ such that $1 \leq i_1 < i_2 < \cdots < i_j \leq k$.

[6]Strict inequality $w_1 < w_2$ simply means $w_1 \leq w_2$ and $w_1 \neq w_2$.

Let us end this section with the following corollary (Given in [HR10, 10.2.3] without proof), it will be needed for the proof of Lemma III.8.0.3.

**Corollary** III.3.0.1. *Let $x, y \in \widetilde{W}$ then*

$$IxIyI \subset \bigsqcup_{z \leq y} IxzI.$$

*Proof.* The corollary will follow using Lemma III.3.0.1 by induction on the size of a minimal word for $y$. Let $y = s_1 \in \mathcal{S}$, we have

$$IxIs_1I = \begin{cases} Ixs_1I, & \text{if } x < xs_1, \\ IxI \sqcup Ixs_1I, & \text{if } xs_1 < x. \end{cases}$$

So it suffices to take $z \in \{y\}$ if $x < xs_1$ and $z \in \{s_1y, y\}$ if $xs_1 < x$. Let $y = s_1 \cdots s_r$ a reduced word for $y$. Put $y' = \prod_{i=2}^{r} s_i$, we then have

$$\begin{aligned} IxIyI = IxIs_1y'I &\overset{\text{Lemma III.3.0.1}}{=} IxIs_1Iy'I \\ &= \begin{cases} Ixs_1Iy'I, & \text{if } x < xs_1, \\ IxIy'I \sqcup Ixs_1Iy'I, & \text{if } xs_1 < x \end{cases} \\ &\subset \begin{cases} \bigsqcup_{z \leq y'} Ixs_1zI, & \text{if } x < xs_1, \\ \bigsqcup_{z \leq y'} IxzI \cup \bigsqcup_{z \leq y'} Ixs_1zI, & \text{if } xs_1 < x \end{cases}. \end{aligned}$$

where the last inclusion is just the recursion hypothesis. Now if $z \leq y'$ then clearly $z < y$ and $s_1z \leq y$, hence in both cases we have

$$IxIyI \subset \bigsqcup_{z \leq y} IxzI. \quad \square$$

## III.4 Dominance in $\Lambda_M$

We have seen in Lemma III.2.0.2 the group $\Lambda_M = M/M_1$, where $M_1$ is by definition the kernel of the surjective Kottwitz homomorphism $\kappa_M$. We then have a canonical isomorphism (See §II.3.9.2)

$$\Lambda_M = M/M_1 \simeq \kappa_M(M) = X^*(Z(\widehat{\mathbf{M}}))^{\sigma}_{\mathcal{I}}.$$

This shows that $\Lambda_M$ is finitely generated abelian group, with torsion subgroup $(\Lambda_M)_{\text{tor}} = \ker \nu_M / \ker \kappa_M = M^1/M_1$.

**Remark** III.4.0.1. *The group $\Lambda_M$ plays the role of the cocharacters lattice in the split case (e.g. in the Satake isomorphism). If $\mathbf{M}$ is a split torus or if $\mathbf{G}$ is unramified, or semisimple and simply connected then the group $\Lambda_M$ has no torsion, i.e. $M^1 = M_1$ (see Remarks III.15.1.1 and III.15.1.2).*

There exists a natural injective finite-cokernel homomorphism $X_*(\mathbf{S}) \hookrightarrow X^*(Z(\widehat{\mathbf{M}}))^{\sigma}_{\mathrm{In}}$ [Ros15, §2.7], which yields an isomorphisms

$$X_*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{R} \simeq X^*(Z(\widehat{\mathbf{M}}))^{\sigma}_{\mathrm{In}} \otimes_{\mathbb{Z}} \mathbb{R} \simeq \Lambda_M \otimes_{\mathbb{Z}} \mathbb{R}.$$

The map $\nu_M \colon M \to X_*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{R}$ (Lemma II.3.2.2) identifies $\Lambda_M/(\Lambda_M)_{\mathrm{tor}} = M/M^1$ with a lattice $\underline{\Lambda}_M$ in $X_*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{R}$, and $\mathrm{rank}(\Lambda_M/(\Lambda_M)_{\mathrm{tor}}) = \dim_{\mathbb{R}} X_*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{R}$. By the above isomorphisms, the extended geometric apartment $\mathbb{A}(\mathbf{G},\mathbf{S})_{\mathrm{ext}}$ acquires the structure of an affine space over $\Lambda_M \otimes_{\mathbb{Z}} \mathbb{R} = \underline{\Lambda}_M \otimes_{\mathbb{Z}} \mathbb{R}$, thus one gets an embedding

$$\underline{\Lambda}_M \to \mathbb{A}_{\mathrm{ext}}(\mathbf{G},\mathbf{S}), \quad v \mapsto \nu(v) + a_\circ.$$

Define[7],

$$M^{\pm} := \{m \in M \colon (\alpha+0)(\nu(m)+a_\circ) = \langle \nu_M(m), \alpha\rangle \geq 0,\ \forall \alpha \in \Phi^{\pm}_{\mathrm{red}}\}.$$

We call $M^+$ (resp. $M^-$) the set of dominant (resp. antidominant) elements of $M$. Let $\Lambda^{\pm}_M \subset \Lambda_M$ and $\underline{\Lambda}^{\pm}_M \subset \underline{\Lambda}_M$ denote the images of $M^{\pm}$ by the natural projections $M \twoheadrightarrow \Lambda^{\pm}_M \twoheadrightarrow \underline{\Lambda}^{\pm}_M$. In other words

$$\underline{\Lambda}^{\pm}_M := \big((a_\circ + \underline{\Lambda}_M) \cap \mathcal{C}^{\pm}\big) - a_\circ \subset \underline{\Lambda}_M,$$

where $\mathcal{C}^{\pm}$ are the two vectorial chambers

$$\mathcal{C}^{\pm} := \{a \in \mathbb{A}_{\mathrm{ext}}(\mathbf{G},\mathbf{S}) \colon (\alpha+0)(a) = \alpha(a-a_\circ) = \langle a - a_\circ, \alpha\rangle \geq 0,\ \forall \alpha \in \Phi^{\pm}_{\mathrm{red}}\}.$$

**Remark** III.4.0.2. *An element $m \in M$ is in $M^+$ if and only if $\nu_N(m)(a_\circ) = \nu(m) + a_\circ \in \overline{\mathcal{C}}^+$ (topological closure), and accordingly also $\nu_N(m)(\mathfrak{a}) \subset \mathcal{C}^+$ since by definition $\mathfrak{a} \subset \mathcal{C}^+$.*

**Lemma** III.4.0.1. *We have the following properties regarding $\Lambda_M$:*

1. *The elements of $\Lambda^-_M$ form a set of representatives for the orbits of the action of the Weyl group $W$ on $\Lambda_M$.*
2. *If $m_1, m_2, \ldots, m_k \in \Lambda_M$, then there exists $m_\circ \in \Lambda^-_M$ such that $m_\circ + m_i \in \Lambda^-_M$[8] for all $1 \leq i \leq k$.*
3. *If $m_1, m_2 \in \Lambda^-_M$, then $\ell(m_1+m_2) = \ell(m_1) + \ell(m_2)$.*

[7] Reminder: the notation $\alpha + 0$ was introduced in definition II.3.4.1.
[8] We will denote the operation on $\Lambda_M$ additively.

*For every $w \in W$, we have:*

4. $\ell(mw) = \ell(w) + \ell(m)$ *if* $m \in \Lambda_M^-$.
5. $\ell(w(m)) = \ell(wmw^{-1}) = \ell(m)$ *for all* $m \in \Lambda_M$.

*Proof.* The first property is [HV15, §6.3 Lemma]. The rest can be found in [Ros15, Lemma 5.2.1]. □

**Remark** III.4.0.3. *Of course the above lemma remains valid if we replace $\Lambda_M^-$ by $\Lambda_M^+$.*

We close this subsection with:

**Proposition** III.4.0.1 (Cartan decomposition for $K$). *The map $G \to K\backslash G/K$ defined by $m \mapsto KmK$, induces a bijection $\Lambda_M^- \simeq K\backslash G/K \simeq \Lambda_M^+$.*

*Proof.* This is a consequence of Lemma III.4.0.1 (1), and the Bruhat decomposition for $G$ in proposition III.2.0.1. □

## III.5 Relative Hecke algebras

We attach for any ring $R$ the $R$-module $\mathcal{C}_c(G, R)$ of locally constant and compactly supported functions $f\colon G \to R$. Let $H$ be any open compact subgroup of $G$. The group $G$ has a unique left invariant measure $\mu_H$ normalized by $H$ on $\mathbb{Q}$ [Vig96, §2.4]:

$$\mathcal{C}_c(G, \mathbb{Q}) \longrightarrow \mathbb{Q}, \quad f \longmapsto \int_G f(g) d\mu_H(g),$$

such that

$$\int_G \mathbf{1}_H(g) d\mu_H(g) = 1.$$

The vector space $\mathcal{C}_c(G, \mathbb{Q})$ acquire the structure of a $\mathbb{Q}$-algebra without a unit, when endowed with the convolution product with respect to $\mu_H$:

$$f *_H f' \colon x \mapsto \int_G f(g) f'(g^{-1}x) d\mu_H(g) \quad (f, f' \in \mathcal{C}_c(G, \mathbb{Q})).$$

**Remark** III.5.0.1. *The above expression of the convolution can be rewritten*

$$\begin{aligned} f *_H f'(x) &= \int_G f(g) f'(g^{-1}x) d\mu_H(g) \\ &= \int_G f(xg) f'(g^{-1}) d\mu_H(xg) && (g \mapsto xg) \\ &= \int_G f(xg) f'(g^{-1}) d\mu_H(g). \end{aligned}$$

*This shows that for $f, f' \in \mathcal{C}_c(G,\mathbb{Q})$, if $f$ is $X$-invariant on the left and $f'$ is $Y$-invariant on the right, then $f *_H f'$ is also $X$-invariant on the left and $Y$-invariant on the right.*

The group $G$ acts on the underlying $\mathbb{Q}$-vector space of this algebra by translation, on the left and on the right as follows:

$$(G\times G)\times \mathcal{C}_c(G,\mathbb{Q}) \longrightarrow \mathcal{C}_c(G,\mathbb{Q})$$

$$((g,g'),f) \longmapsto ((g,g')\cdot f\colon x\mapsto f(g^{-1}xg')).$$

**Lemma** III.5.0.1. *Let $X, Y$ be two open compact subgroups of $G$. For any $g, h \in G$, one has*

$$(i) \qquad \mathbf{1}_{gY} *_H \mathbf{1}_{hX} = |Y\cap hXh^{-1}|_H \mathbf{1}_{gYhX}$$

$$(ii) \qquad \mathbf{1}_{gY} *_H \mathbf{1}_{XhX} = \frac{|Y\cap X|_H}{|X\cap hXh^{-1}|_H}\mathbf{1}_{gYX} *_H \mathbf{1}_{hX},$$

*where, the notation $|\Box|_H$ denotes the volume of $\Box$ with respect to the measure $\mu_H$.*

*Proof.* (i) Observe that the function

$$\mathbf{1}_{gY} *_H \mathbf{1}_{hX}(a) = \int_G \mathbf{1}_{gY}(b)\mathbf{1}_{hX}(b^{-1}a)d\mu_H(b),$$

can only be nonzero on the set $gYhX$. Let $a\in gYhX$ and write it as $a = gyhx$, thus

$$\begin{aligned}\mathbf{1}_{gY} *_H \mathbf{1}_{hX}(a) &= |gY\cap aXh^{-1}|_H = |gY\cap gyhxXh^{-1}|_H\\ &= |gyY\cap gyhXh^{-1}|_H = |Y\cap hXh^{-1}|_H\end{aligned}$$

where, the third equality holds thanks to the left invariance of the measure. Therefore,

$$\mathbf{1}_{gY} *_H \mathbf{1}_{hX} = |Y\cap hXh^{-1}|\mathbf{1}_{gYhX}.$$

In particular, taking $g = 1$ and $Y = X$ we get $\mathbf{1}_X *_H \mathbf{1}_{hX} = |X\cap hXh^{-1}|_H\mathbf{1}_{XhX}$.

(ii) Using (i) we get

$$\begin{aligned}\mathbf{1}_{gY} *_H \mathbf{1}_{XhX} &= \frac{1}{|X\cap hXh^{-1}|_H}(\mathbf{1}_{gY} *_H \mathbf{1}_X) *_H \mathbf{1}_{hX}\\ &= \frac{|Y\cap X|_H}{|X\cap hXh^{-1}|_H}\mathbf{1}_{gYX} *_H \mathbf{1}_{hX}.\end{aligned}$$ $\Box$

**Example** III.5.0.1. *For $g, g' \in G$, write $Hg'H$ as disjoint union $\sqcup_{h'} h'g'H$, then by lemma above:*

$$\mathbf{1}_{gH} *_H \mathbf{1}_{Hg'H} = \mathbf{1}_{gHg'H} = \mathbf{1}_{\sqcup_{h'}gh'g'H} = \sum_{h'}\mathbf{1}_{gh'g'H}.$$

*If also* $HgH = \sqcup_h hgH$, *then*

$$\mathbf{1}_{HgH} *_H \mathbf{1}_{Hg'H} = (\sum_h \mathbf{1}_{hgH}) *_I \mathbf{1}_{Hg'H} = \sum_{h,h'} \mathbf{1}_{hgh'g'H}.$$

Now, we associate to the pair $(G,H)$ different $\mathbb{Z}$-modules.

**Definition** III.5.0.1. *We define $\mathcal{C}_c(G/H,\mathbb{Z})$ to be the $\mathbb{Z}$-module of compactly supported functions $f\colon G \to \mathbb{Z}$ which are $H$-invariant on the right. It has the following canonical basis $\{\mathbf{1}_{gH}\colon g \in G/H\}$. In addition, the group left action of $G$ on $\mathcal{C}_c(G,\mathbb{Q})$ as defined above, restricts to a left action on $\mathcal{C}_c(G/H,\mathbb{Z})$. We also define $\mathcal{C}_c(G /\!\!/ H,\mathbb{Z}) \subset \mathcal{C}_c(G/H,\mathbb{Z})$ to be the $\mathbb{Z}$-algebra*[9] *of functions $f\colon G \to \mathbb{Z}$, that are also $H$-invariant on the left. We call $\mathcal{C}_c(G /\!\!/ H,\mathbb{Z})$ the Hecke algebra relative to $H$ and denote it $\mathcal{H}_H(\mathbb{Z})$. Likewise, for every commutative ring $A$, we define the relative $A$-algebra obtained by base change*

$$\mathcal{H}_H(A) = \mathcal{C}_c(G /\!\!/ H,\mathbb{Z}) = \mathcal{H}_H(\mathbb{Z}) \otimes_{\mathbb{Z}} A.$$

The Hecke algebra $\mathcal{H}_H(\mathbb{Z})$ is a free $\mathbb{Z}$-algebra, it has the canonical basis $\{\mathbf{1}_{HgH}\colon g \in HgH\}$.

**Proposition** III.5.0.1. *The following map*[10]

$$\begin{aligned} \mathcal{H}_H(\mathbb{Z}) &\xrightarrow{\simeq} \operatorname{End}_{\mathbb{Z}[G]} \mathcal{C}_c(G/H,\mathbb{Z})^{\mathrm{opp}} \\ h &\longmapsto e_h\colon f \mapsto f *_H h, \end{aligned}$$

*is an isomorphism of rings.*

*Proof.* We first prove that the above map is an isomorphism of $\mathbb{Z}$-modules. The injectivity being clear, we prove surjectivity. Any $G$-equivariant endomorphism $e$ of $\mathcal{C}_c(G/H,\mathbb{Z})$ is uniquely defined by $e(\mathbf{1}_H)$. Since $\mathbf{1}_H$ is $H$-invariant on the left, $e(\mathbf{1}_H)$ must also be $H$-invariant on the left. This shows that $e$ is the image of the Hecke element $e(\mathbf{1}_H)$.

Finally, for any $h_1, h_2 \in \mathcal{H}_H(\mathbb{Z})$, we have

$$\begin{aligned} e_{h_1 *_H h_2}(\mathbf{1}_H) &= \mathbf{1}_H *_H (h_1 *_H h_2) \\ &= (\mathbf{1}_H *_H h_1) *_H h_2 \\ &= e_{h_1}(\mathbf{1}_H) *_H h_2 \\ &= e_{h_2} \circ e_{h_1}(\mathbf{1}_H) \end{aligned}$$

[9] By Remark III.5.0.1 and Example III.5.0.1, we see that $\mathcal{C}_c(G /\!\!/ H,\mathbb{Z}) \subset \mathcal{C}_c(G /\!\!/ H,\mathbb{Q})$ is stable under $*_H$.

[10] The notation $\operatorname{End}_{\mathbb{Z}[G]} \mathcal{C}_c(G/H,\mathbb{Z})$ denotes the ring of $G$-equivariant endomorphisms of $\mathcal{C}_c(G/H,\mathbb{Z})$, and the superscript opp indicates the opposite ring.

As we previously said, elements of $\mathrm{End}_{\mathbb{Z}[G]}\,\mathcal{C}_c(G/H,\mathbb{Z})$ are uniquely determined by the image they give to $\mathbf{1}_H$, this shows that $e_{h_1 *_H h_2} = e_{h_2}\circ e_{h_1}$. This ends the proof of the lemma. $\square$

**Remark** III.5.0.2. *The map*

$$\mathcal{H}_H(\mathbb{Z}) \xrightarrow{\ \simeq\ } \mathcal{H}_H(\mathbb{Z})$$

$$h \longmapsto h^\vee : g \mapsto h(g^{-1}),$$

*is an involution, and for any $h_1, h_2 \in \mathcal{H}_H(\mathbb{Z})$ we have*

$$(h_1 *_H h_2)^\vee = h_2^\vee *_H h_1^\vee. \quad \square$$

The following lemma is meant to clarify the multiplicative structure of the relative Hecke algebra $\mathcal{H}_H(\mathbb{Z})$.

**Lemma** III.5.0.2. *For $g, g' \in G$, we have*

$$\mathbf{1}_{HgH} *_H \mathbf{1}_{Hg'H} = \sum_{g''\in C_{g,g'}} c(g,g',g'')\mathbf{1}_{Hg''H}, \quad c(g,g',g'') = \left|HgH \cap g''Hg'^{-1}H\right|_H$$

*where, $C_{g,g'}$ denotes a set of representatives for $H\backslash HgHg'H/H$.*

*Proof.* First, we know that the function $\mathbf{1}_{HgH} *_H \mathbf{1}_{Hg'H}$ is $H$-biinvariante (Remark III.5.0.1), therefore, using the canonical basis of $\mathcal{H}_H(\mathbb{Z})$, it can be written as follows

$$\mathbf{1}_{HgH} *_H \mathbf{1}_{Hg'H} = \sum_{g''\in C_{g,g'}} c(g,g',g'')\mathbf{1}_{Hg''H}, \quad c(g,g',g'') \in \mathbb{N}$$

for some finite index set $C_{g,g'}$ and integral coefficients $c(g,g',g'') \neq 0$, for each $g'' \in C_{g,g'}$.

Secondly, the following integral

$$\mathbf{1}_{HgH} *_H \mathbf{1}_{Hg'H}(a) = \int_G \mathbf{1}_{HgH}(b)\mathbf{1}_{Hg'H}(b^{-1}a)d\mu_H(b),$$

is nonzero only if $a \in HgHg'H$. Therefore, if $Hg''H \subset HgHg'H$, we have

$$c(g,g',g'') = \mathbf{1}_{HgH} *_H \mathbf{1}_{Hg'H}(g'') = \left|HgH \cap g''Hg'^{-1}H\right|_H \in \mathbb{Z}_{\geq 0}.$$

For each double coset $Hg''H \subset HgHg'H$, write $g'' = gh''g'$ for some $h'' \in H$, thus

$$c(g,g',g'') = \left|HgH \cap gh''g'Hg'^{-1}H\right|_H \geq \left|gH\right|_H = 1.$$

This shows that $C_{g,g'}$ is indeed a set of representatives for $H\backslash HgHg'H/H$. $\square$

**Remark** III.5.0.3. *For any $g, g' \in G$, we have*

$$\begin{aligned}
\int_G \mathbf{1}_{HgH} *_H \mathbf{1}_{Hg'H}(a) d\mu_H(a) &= \int_G \int_G \mathbf{1}_{HgH}(b)\mathbf{1}_{Hg'H}(b^{-1}a) d\mu_H(a) d\mu_H(b) \\
&= \int_G \int_G \mathbf{1}_{HgH}(b)\mathbf{1}_{Hg'H}(b^{-1}a) d\mu_H(b) d\mu_H(a) \qquad (\textit{Fubini}) \\
&= \int_G \left( \int_G \mathbf{1}_{Hg'H}(b^{-1}a) d\mu_H(a) \right) \mathbf{1}_{HgH}(b) d\mu_H(b) \\
&= \int_G |bHg'H|_H \, \mathbf{1}_{HgH}(b) d\mu_H(b) \\
&= |Hg'H|_H \int_G \mathbf{1}_{HgH}(b) d\mu_H(b) \\
&= |HgH|_H \, |Hg'H|_H \, .
\end{aligned}$$

*This suggest (and shows) that the linear functional*

$$d_H \colon \mathcal{H}_H(\mathbb{Z}) \longrightarrow \mathbb{Z},$$

*defined on the canonical basis elements by $\mathbf{1}_{HgH} \longmapsto |HgH|_H$, is an isomorphism of rings. Consequently, we see that*

$$\mathbf{1}_{HgH} \in \mathcal{H}_H(\mathbb{Z})^\times \iff d_H\left(\mathbf{1}_{HgH}\right) = 1 \iff g \in N_G(H).$$

**Lemma** III.5.0.3. *Let $H' \subset H$ be two open compact subgroups of $G$ (e.g. $H' = I$ and $H = K$). The function $e_H := |H|_{H'}{}^{-1}\mathbf{1}_H$ is an idempotent of the relative Hecke algebra $\mathcal{H}_{H'}(\mathbb{Q})$, moreover we have a natural isomorphism of rings*

$$(e_H *_{H'} \mathcal{H}_{H'}(\mathbb{Q}) *_{H'} e_H, +, *_{H'}) \xrightarrow{\simeq} (\mathcal{H}_H(\mathbb{Q}), +, *_H), \ .$$

*Proof.* 1. Idempotence: By Lemma III.5.0.1, we have $\mathbf{1}_H *_{H'} \mathbf{1}_H = |H|_{H'}\mathbf{1}_H$, this shows that $e_H$ si indeed an idempotent.

2. Let us compare the two associated measures $\mu_H$ and $\mu_{H'}$. We have

$$\begin{aligned}
1 = |H|_H &= \int_G \mathbf{1}_H(g) d\mu_H(g) \\
&= \sum_{a \in H/H'} \int_G \mathbf{1}_{aH'}(g) d\mu_H(g) \\
&= \sum_{a \in H/H'} |H'|_H
\end{aligned}$$

Hence $[H : H'] = |H|_{H'} = |H'|_H^{-1}$, and $d\mu_{H'} = [H : H'] d\mu_H(g)$ by uniqueness of the Haar measure.

3. Using Lemma III.5.0.1, we have for any $g \in G$

$$\begin{aligned} e_H *_{H'} \mathbf{1}_{H'gH'} *_{H'} e_H &= \frac{1}{[H:H']^2} \frac{|H \cap gHg^{-1}|_{H'}}{|H' \cap gH'g^{-1}|_{H'}} \mathbf{1}_{HgH} \\ &= \frac{1}{[H:H']} \frac{|H \cap gHg^{-1}|}{|H|_{H'}} \frac{|H'|_{H'}}{|H' \cap gH'g^{-1}|} \mathbf{1}_{HgH} \\ &= \frac{1}{[H:H']} \frac{|H'gH'|_{H'}}{|HgH|_H} \mathbf{1}_{HgH} \\ &\overset{\text{Remark III.5.0.3}}{=} \frac{1}{[H:H']} \frac{d_{H'}(g)}{d_H(g)} \mathbf{1}_{HgH}. \end{aligned}$$

Consider the following isomorphism of $\mathbb{Q}$-modules

$$(e_H *_{H'} \mathcal{H}_{H'}(\mathbb{Q}) *_{H'} e_H, +, *_{H'}) \xrightarrow{\simeq} (\mathcal{H}_H(\mathbb{Q}), +, *_H),$$

defined on the canonical basis elements

$$\mathbf{1}_{HgH} \longmapsto \frac{1}{[H:H']} \mathbf{1}_{HgH}.$$

and since

$$\frac{1}{[H:H']} \mathbf{1}_{HgH} *_H \frac{1}{[H:H']} \mathbf{1}_{Hg'H} = \frac{1}{[H:H']} \mathbf{1}_{HgH} *_{H'} \mathbf{1}_{Hg'H} \quad (\forall g, g' \in G),$$

this is actually an isomorphism of rings. $\square$

**Remark** III.5.0.4. *The identification of Lemma III.5.0.3 remains valid, by the same proof, if we replace $\mathbb{Q}$ by any subring in which $[H:H']$ is invertible.*

## III.6 Special–Hecke and Iwahori–Hecke algebras

**Definition** III.6.0.1. *The Hecke algebra $(\mathcal{H}_K(\mathbb{Z}), +, *_K)$ relative to the maximal special parahoric subgroup $K$, will be called the special–Hecke algebra (sometimes the spherical Hecke algebra, or the Hecke algebra in short). The Hecke algebra $(\mathcal{H}_I(\mathbb{Z}), +, *_I)$ relative to the Iwahori subgroup $I$, will be called the Iwahori–Hecke algebra.*

Using Proposition III.4.0.1, one can exhibit a natural $\mathbb{Z}$-basis for $\mathcal{H}_K(\mathbb{Z})$ as follows

$$\{h_m := \mathbf{1}_{KmK} \text{ for } m \in \Lambda_M^-\}.$$

Similarly, according to Proposition III.2.0.1, the following set forms a $\mathbb{Z}$-basis for $\mathcal{H}_I(\mathbb{Z})$

$$\{i_w := \mathbf{1}_{IwI} \text{ for } w \in \widetilde{W}\}.$$

Recall that $\mathcal{S} \subset N_1/M_1$ corresponds to the set of orthogonal reflections with respect to the walls of the fixed alcove $\mathfrak{a}$ (Using the identification $N_1/M_1 \simeq W_{\text{aff}}$, see §III.1).

**Theorem** III.6.0.1 (The Iwahori–Matsumoto Presentation). *The Iwahori–Hecke ring $\mathcal{H}_I(\mathbb{Z})$ is the free $\mathbb{Z}$-module with basis $(i_w)_{w\in\widetilde{W}}$ endowed with the unique ring structure satisfying*

- *The braid relations:*
$$i_w i_{w'} = i_{ww'} \text{ if } w, w' \in \widetilde{W} \text{ such that } \ell(w) + \ell(w') = \ell(ww').$$
- *The quadratic relations:*
$$i_s^2 = q_s i_1 + (q_s - 1) i_s \text{ if } s \in \mathcal{S},$$
*where, $q_w := [IwI : I]$ denotes the number of left $I$-cosets in $IwI$ for $w \in \widetilde{W}$.*

*Proof.* This is [Vig16, Theorem 2.1]. The Braid relation follows from Lemma III.3.0.1 and Remark III.3.0.1. □

**Remark** III.6.0.1. *The above Iwahori–Matsumoto presentation yields the following consequences:*

- *Since for any $z, z' \in \widetilde{\Omega}$ we have $\ell(z) = \ell(z') = 0 = \ell(zz')$, by the braid relations,*
$$i_z *_I i_{z'} = i_{zz'},$$
*hence, the $\mathbb{Z}$-linear map $z \mapsto i_z$ embeds the group algebra $\mathbb{Z}[\widetilde{\Omega}]$ into $\mathcal{H}_I(\mathbb{Z})$. In particular, $i_z \in \mathcal{H}_I(\mathbb{Z})^\times$ for any $z \in \widetilde{\Omega}$, with inverse $i_{z^{-1}}$.*
- *Recall that $(N_1/M_1, \mathcal{S}) \simeq (W_{\text{aff}}, \mathcal{S}(\mathfrak{a}))$ is a Coxeter system (Theorem III.1.0.1). Let $w \in W_{\text{aff}}$ and $w = s_1 s_2 \cdots s_{\ell(w)}$ a reduced expression, then using Lemma III.3.0.1 one shows that*
$$q_w = q_{s_1} q_{s_2} \cdots q_{s_{\ell(w)}}.$$
*Let $z \in \widetilde{\Omega}_G$, then (by definition of $\widetilde{\Omega}_G$) $z$ normalizes $I$ and $q_z = 1$, thus*
$$q_? : \widetilde{W} \overset{\text{Lemma III.2.0.1}}{=} W_{\text{aff}} \rtimes \widetilde{\Omega}_G \to \mathbb{N}$$
*factors through $W_{\text{aff}}$.*
- *For any $w \in W_{\text{aff}}$ the intger $q_w$ is actually a power of $q$. By the previous remark, it suffices to show it for all affine reflections $s \in \mathcal{S}$. Using the bijection $IsI/I \simeq I/(I \cap sIs)$, and writing the latter as quotient of two affine root groups [Vig16, Proposition 3.23.] one then shows the claim, see Corollary 3.31* loc. cit..

- *Note that* $i_s \in \mathcal{H}_I(\mathbb{Z}[q^{\pm 1}])^\times$ *for all* $s \in \mathcal{S}$ *with inverse* $i_s^{-1} = q_s^{-1}(i_s - q_s + 1)$, *hence by the braid relation* $i_w \in \mathcal{H}_I(\mathbb{Z}[q^{\pm 1}])^\times$ *for all* $w \in \widetilde{W}$. *Indeed, let* $w = s_1 \cdots s_{\ell(w)} z$, *with* $s_1 \cdots s_{\ell(w)}$ *a reduced word in* $W_{\mathrm{aff}}$ *and* $z \in \widetilde{\Omega}$. *By the Braid relation we have* $i_w = i_{s_1} \cdots i_{s_{\ell(w)}} i_z$, *accordingly*

$$\begin{aligned} i_w^{-1} &= i_z^{-1} i_{s_{\ell(w)}}^{-1} \cdots i_{s_1}^{-1} \\ &= i_z^{-1} \frac{1}{q_{s_{\ell(w)}}}(i_{s_{\ell(w)}} - q_{s_{\ell(w)}} + 1) \cdots \frac{1}{q_{s_1}}(i_{s_1} - q_{s_1} + 1) \\ &= \prod_{i=1}^{\ell(w)} \frac{1}{q_{s_i}} i_{z^{-1}}(i_{s_{\ell(w)}} - q_{s_{\ell(w)}} + 1) \cdots (i_{s_1} - q_{s_1} + 1) \\ &= \frac{1}{q_w} i_{z^{-1}}(i_{s_{\ell(w)}} - q_{s_{\ell(w)}} + 1) \cdots (i_{s_1} - q_{s_1} + 1). \end{aligned}$$

- *If we do not include the inverse of* $q$ *in the coefficients ring, we still have*

$$i_w i_w^* = i_w^* i_w = q_w, \quad \text{in } \mathcal{H}_I(\mathbb{Z}),$$

*where,* $i_w^* := (i_{s_{\ell(w)}} - q_{s_{\ell(w)}} + 1) \cdots (i_{s_1} - q_{s_1} + 1)$.

- *For all* $w \in W_{\mathrm{aff}}$ *and* $z \in \widetilde{\Omega}$, *let* $w = s_1 \cdots s_{\ell(w)}$ *be a reduced expression, hence by the Braid relation again we get*

$$\begin{aligned} i_w i_z &= i_{wz} \\ &= i_{s_1} \cdots i_{s_{\ell(w)}} i_z \\ &= i_z i_{z^{-1} s_1 z} \cdots i_{z^{-1} s_{\ell(w)} z} \\ &= i_z i_{z^{-1} w z}. \end{aligned}$$

□

## III.7 Iwahori decompositions

We now give an Iwahori decomposition for the special parahoric subgroup $K$. Note that we are no longer in the unramified case where we could have pulled up the Bruhat decomposition for the residue field of $F$.

**Proposition** III.7.0.1 (Iwahori decomposition of $K$ - I). *We have the decomposition*

$$K = \bigsqcup_{w \in W} IwI.$$

*Proof.* By the Bruhat decomposition $G = INI$ (Proposition III.2.0.1) every element $k \in K$ can be written in the form $ini'$, for some $n \in N \cap K$ and $i, i' \in I$. It is then clear that we have $K = I(N \cap K)I$, hence

$$K = \bigcup_{n \in [n] \in N \cap K / N \cap I} InI.$$

**Lemma** III.7.0.1.

$$M_1 = M \cap K = M \cap I = N \cap I.$$

*Proof of the lemma.* The first two equalities are a particular case of Lemma II.3.9.2 by taking $\mathcal{F}$ to be $a_\circ$ then $\mathfrak{a}$. For the second equality, let $n \in N \cap I$. By the Iwahori factorization of $I = (I \cap U^+)(I \cap U^-)M_1$ (Corollary II.3.9.2), there exists $u \in (I \cap U^+)(I \cap U^-)$ and $m \in M_1$ such that $n = um$. Thus $n = m$ since $nm^{-1} \in N \cap U^+U^- = \{1\}$ [BT65, 5.15]. This proves $N \cap I \subset M_1$. The other inclusion is clear since $M_1 \subset I$ by Iwahori factorization of the latter. □

Using Lemma III.7.0.1 we get

$$K = \bigcup_{n \in [n] \in N \cap K / N \cap I} InI = \bigcup_{n \in [n] \in N \cap K / M \cap K} InI = \bigcup_{w \in W} IwI.$$

It remains to prove that the union is disjoint. This follows from the fact that the map $N \to I \backslash G / I \simeq \widetilde{W}$ given by $n \mapsto InI$ has kernel $M_1$ since by definition $\widetilde{W} = N/M_1$ (Proposition III.2.0.1). This means that the union is disjoint in $G$ and hence also in $K$. □

**Remark** III.7.0.1. *The preceding proof may be applied more generally. Let $\mathcal{F}$ be a facet lying in the closure of $\mathfrak{a}$. Let $K_{\mathcal{F}}$ be its associated parahoric subgroup (Definition II.3.9.1). Since $\mathcal{F} \subset \bar{\mathfrak{a}}$ we have $I \subset K_{\mathcal{F}}$. We have*

$$\begin{aligned}
K_{\mathcal{F}} &= I(N \cap K_{\mathcal{F}})I && \textit{Proposition III.2.0.1} \\
&= I(N \cap K_{\mathcal{F}} / N \cap I)I \\
&= \bigcup_{n \in N \cap K_{\mathcal{F}} / N \cap I} InI \\
&= \bigcup_{n \in N \cap K_{\mathcal{F}} / M \cap K_{\mathcal{F}}} InI && \textit{Lemma III.7.0.1} \\
&= \bigcup_{n \in N \cap K_{\mathcal{F}} / M_1} InI && \textit{Lemma II.3.9.2} \\
&= \bigsqcup_{n \in N \cap K_{\mathcal{F}} / M_1} InI && \textit{Proposition III.2.0.1.}
\end{aligned}$$

*Here, the quotient $\widetilde{W}^{K_{\mathcal{F}}} := N \cap K_{\mathcal{F}} / M_1$ identifies with a finite Coxeter subgroup $W_{\mathrm{aff}}^{\mathcal{F}} \subset W_{\mathrm{aff}}$*

*(Definition II.3.5.2). We may then rewrite the above decomposition as follows*

$$K_{\mathcal{F}} = \bigsqcup_{w \in \widetilde{W}^{K_{\mathcal{F}}}} IwI.$$

**Proposition** III.7.0.2 (Iwahori decomposition of $K$ - II). *We have the decomposition*

$$K = \bigsqcup_{w \in W} (B \cap K)wI.$$

*Proof.* We have the following two statement:

1. By [HV15, Theorem §6.5] one has $B \cap K = M_1(U^+ \cap K)$.
2. Let us show that $U^+ \cap K = I^+$. Recall that Corollary II.3.9.2 gives an Iwahori factorization $I = I^+ M_1 I^-$ with
$$I^+ := I \cap U^+ = \prod^{\prec}_{\alpha \in \Phi^+ \cap \Phi_{\mathrm{red}}} U_{\alpha+0}, \quad \text{and} \quad I^- := I \cap U^- = \prod^{\prec}_{\alpha \in \Phi^- \cap \Phi_{\mathrm{red}}} U_{\alpha+n_\alpha^{-1}}$$
for a fixed ordering $\prec$ of the set $\Phi_{\mathrm{red}}$ such that the positive roots occur before the negative ones. Using the second part of Corollary II.3.9.2, we obtain
$$U^+ \cap K = \prod^{\prec}_{\alpha \in \Phi^+ \cap \Phi_{\mathrm{red}}} U_{\alpha+0} = I^+.$$

Combining the above two points and Lemma III.7.0.2 below, we find:

$$IwI = I^+ M_1 I^- wI = (B \cap K)I^- wI = (B \cap K)w(w^{-1}I^- w)I = (B \cap K)wI.$$

Finally, Proposition III.7.0.1 gives the desired decomposition. □

**Lemma** III.7.0.2. *For every $w \in W$ we have $wI^-w^{-1} \subset I$.*

*Proof.* Let $n \in N \cap K$, with image $w$ in $W$. By Lemma II.3.4.1 we have

$$nU_{\alpha+n_\alpha^{-1}}n^{-1} = U_\beta,$$

where, $\beta \colon \mathbb{A}_{\mathrm{red}}(\mathbf{G}, \mathbf{S}) \to \mathbb{R}$ is the affine map $w(\alpha) + n_\alpha^{-1} - w(\alpha)(\nu_N(n)(a_\circ) - a_\circ)$, thus $\beta = w(\alpha) + n_\alpha^{-1}$ since $n \in N \cap K$ fixes $a_\circ$. For a fixed ordering $\prec$ of $\Phi^- \cap \Phi_{\mathrm{red}}$, we get

$$nI^-n^{-1} = \prod^{\prec}_{\alpha \in \Phi^- \cap \Phi_{\mathrm{red}}} nU_{\alpha+n_\alpha^{-1}}n^{-1} = \prod^{\prec}_{\alpha \in \Phi^- \cap \Phi_{\mathrm{red}}} U_{w(\alpha)+n_\alpha^{-1}}.$$

To conclude, recall that for each $\alpha \in \Phi_{\mathrm{red}}$ one has $U_{\alpha+n_\alpha^{-1}} \subset U_{\alpha+0}$[11], and this shows:

$$nI^-n^{-1} \subset \prod^{\prec}_{\alpha \in \Phi^- \cap \Phi_{\mathrm{red}}} U_{w(\alpha)+0} \subset I. \quad \square$$

[11] Recall that the set of subgroups $(U_{\alpha+r})_{r \in \Gamma_\alpha}$ is by definition decreasing.

We end this section by recalling the Iwasawa decomposition and proving two lemmas we will be regularly using in the sequel.

**Proposition** III.7.0.3 (Iwasawa decomposition). *If $\mathbf{P} = \mathbf{L} \ltimes \mathbf{U}_P^+$ is any semi-standard (see II.2.6) parabolic subgroup of $\mathbf{G}$ with Levi factor $\mathbf{L}$ and unipotent radical $\mathbf{U}_P^+$, then $G = PK$ and*

$$P \cap K = (L \cap K)(U_P^+ \cap K).$$

*Proof.* When the parabolic $\mathbf{P}$ is minimal this was given first in [HR10, §9.1]. For semi-standard parabolic subgroups see [HV15, §6.5 Proposition & Theorem]. □

**Lemma** III.7.0.3. *For every $m \in M^-$ we have*

$$m^{-1} I^- m \subset I^- \quad \textit{and} \quad m\, I^+ m^{-1} \subset I^+$$

*Proof.* We have seen a the following factorization of $I$ in term of affine root groups:

$$I^- = I \cap U^- = \prod_{\alpha \in \Phi^- \cap \Phi_{\text{red}}}^{\prec} U_{\alpha + n_\alpha^{-1}}$$

for a fixed ordering $\prec$ of the set $\Phi^-_{\text{red}}$. For any $r \in \Gamma_\alpha$, we have

$$\begin{aligned} mU_{\alpha+r}m^{-1} &= U_{\alpha + r - \alpha(\nu_N(m)(a_\circ) - a_\circ)} && \text{Lemma II.3.4.1} \\ &= U_{\alpha + r - \langle \nu(m), \alpha \rangle} \end{aligned}$$

Hence[12], for all $r \in \Gamma_\alpha$

$$mU_{\alpha+r}m^{-1} \begin{cases} \subset U_{\alpha+r} & \text{if } \langle \nu(m), \alpha \rangle \leq 0 (\Leftarrow m \in M^{\pm} \text{ and } \alpha \in \Phi^{\mp}_{\text{red}} \cap \Phi_{\text{red}}) \\ \supset U_{\alpha+r} & \text{if } \langle \nu(m), \alpha \rangle \geq 0 (\Leftarrow m \in M^{\pm} \text{ and } \alpha \in \Phi^{\pm}_{\text{red}} \cap \Phi_{\text{red}}) \end{cases}$$

and so

$$m^{-1} I^- m = \prod_{\alpha \in \Phi^- \cap \Phi_{\text{red}}}^{\prec} m^{-1} U_{\alpha + n_\alpha^{-1}} m = \prod_{\alpha \in \Phi^- \cap \Phi_{\text{red}}}^{\prec} U_{\alpha + n_\alpha^{-1} + \langle \nu(m), \alpha \rangle}.$$

In particular, if $m$ is antidominant,

$$m^{-1} I^- m \subset \prod_{\alpha \in \Phi^- \cap \Phi_{\text{red}}}^{\prec} U_{\alpha + n_\alpha^{-1}} = I^-.$$

The second inclusion is obtained similarly. □

**Remark** III.7.0.2. *Let $\Omega \subset \mathcal{A}_{\text{ext}}$ be any bounded subset containing a facet $\mathcal{F}$. As in Remark II.3.9.5, consider the open compact subgroup $K_\Omega = P(F^{un})^\circ_\Omega \cap G_1$. The proof*

[12] Recall that, the set of affine root groups $(U_{\alpha+r})_{r \in \Gamma_\alpha}$ is by definition decreasing (§II.3.3).

*above gives then mutatis mutandis a more general result: Using the same remark, we see that the product maps*

$$\prod_{\alpha\in\Phi_{\mathrm{red}}\cap\Phi^{\pm}} U_{\alpha+f_\Omega(\alpha)} \xrightarrow{\sim} U_\Omega^{\pm} = U_\Omega \cap U^{\pm}$$

*are homeomorphisms whatever ordering of the factors we take. Then for all $m \in M^-$ we have*

$$m^{-1}U_\Omega^- m \subset U_\Omega^-, \quad mU_\Omega^+ m^{-1} \subset U_\Omega^+.$$

*If we assume further that $\Omega$ contains the alcove $\mathfrak{a}$, then*

$$M_1 \subset N \cap K_\Omega \subset N \cap K_{\mathfrak{a}} \overset{\textit{Lem. II.3.9.3}}{\subset} M_1.$$

*Therefore, by Remark II.3.9.5 $K_\Omega$ has an Iwahori factorization with respect to $B$ meaning: the product map*

$$U_\Omega^+ \times M_1 \times U_\Omega^- \longrightarrow K_\Omega$$

*is an isomorphism.*

Let $\delta_B$ be the modular function on the fixed minimal parabolic $B$ containing $M$ given by the normalized absolute value of the determinant of the adjoint action on $\operatorname{Lie} U^+$:

$$\delta_B(m) := |\det(\mathrm{Ad}(m)_{\mathrm{Lie}(U^+)})|_F, \quad \forall m \in M$$

where $|\cdot|_F$ is the fixed normalized absolute value of $F$.

**Remark** III.7.0.3. *Let $\Omega \subset \mathcal{A}_{\mathrm{ext}}$ be any bounded subset containing the alcove $\mathfrak{a}$. An immediate consequence of the above remark is that for any pair $m_1, m_2 \in M^{\pm}$*[13] *we have $K'm_1K'm_2K' = K'm_1m_2K'$. For example, if both are antidominant then*

$$\begin{aligned} K_\Omega m_1 K_\Omega m_2 K_\Omega &= K_\Omega m_1 U_\Omega^+ M_1 U_\Omega^- m_2 K_\Omega \\ &= K_\Omega m_1 U_\Omega^+ M_1 m_1^{-1}(m_1m_2)m_2^{-1} U_\Omega^- m_2 K_\Omega \\ &= K_\Omega m_1 m_2 K_\Omega, \end{aligned}$$

*If they were dominant, the same argument holds using this time the decomposition $K_\Omega = U_\Omega^- M_1 U_\Omega^+$ instead. Now, using Lemma III.5.0.2, we see that for any $m_1, m_2 \in M$ both dominant (resp. antidominant) we have*

$$\mathbf{1}_{K_\Omega m_1 K_\Omega} *_{K_\Omega} \mathbf{1}_{K_\Omega m_2 K_\Omega} = \left|K_\Omega m_1 K_\Omega \cap m_1 m_2 K_\Omega m_2^{-1} K_\Omega\right|_{K_\Omega} \mathbf{1}_{K_\Omega m_1 m_2 K_\Omega}.$$

*When $m_1, m_2 \in M^-$, we claim that $\left|K_\Omega m_1 K_\Omega \cap m_1 m_2 K_\Omega m_2^{-1} K_\Omega\right|_{K_\Omega} = 1$.*

[13] Meaning, both are dominant or both antidominant.

*Proof.* Observe that

$$\begin{aligned} K_\Omega m_1 K_\Omega \cap m_1 m_2 K_\Omega m_2^{-1} K_\Omega &= U_\Omega^+ m_1 K_\Omega \cap m_1 m_2 U_\Omega^- m_2^{-1} K_\Omega \\ &= (U_\Omega^+ m_1 \cap m_1 m_2 U_\Omega^- m_2^{-1} K_\Omega) K_\Omega \cap m_1 m_2 U_\Omega^- m_2^{-1} K_\Omega \\ &= (U_\Omega^+ m_1 \cap m_1 m_2 U_\Omega^- m_2^{-1} K_\Omega) K_\Omega \\ &= (U_\Omega^+ \cap m_1 m_2 U_\Omega^- m_2^{-1} K_\Omega m_1^{-1}) m_1 K_\Omega \end{aligned}$$

now, using (III.7.0.2) and the Iwahori factorization, we compute the intersection between the parentheses

$$U_\Omega^+ \cap m_1 m_2 U_\Omega^- m_2^{-1} K_\Omega m_1^{-1} = U_\Omega^+ \cap M_1 m_1 m_2 U_\Omega^- m_2^{-1} U_\Omega^+ m_1^{-1}$$

Let $u \in U_\Omega^+ \cap m_1 m_2 U_\Omega^- m_2^{-1} K_\Omega m_1^{-1}$, and write it as $u = m_0 u_- u_+$, with $m_0 \in M_1$, $u_- \in m_1 m_2 U_\Omega^- (m_1 m_2)^{-1}$ and $u_+ \in m_1 U_\Omega^+ m_1^{-1}$, this implies

$$\underbrace{(u u_+^{-1})}_{\in U^+} \underbrace{u_-^{-1}}_{\in U^-} = m_0 \in U^+ U^- \cap N$$

But $U^+ U^- \cap N = \{1\}$ [BT65, 5.15], hence $u = u_+$, $u_- = 1$ and $m_0 = 1$, which shows

$$U_\Omega^+ \cap m_1 m_2 U_\Omega^- m_2^{-1} K_\Omega m_1^{-1} = m_1 U_\Omega^+ m_1^{-1},$$

and consequently

$$K_\Omega m_1 K_\Omega \cap m_1 m_2 K_\Omega m_2^{-1} K_\Omega = m_1 K_\Omega.$$

So in conclusion, for $m_1, m_2 \in M^-$ one has

$$\mathbf{1}_{K_\Omega m_1 K_\Omega} *_{K_\Omega} \mathbf{1}_{K_\Omega m_2 K_\Omega} = \mathbf{1}_{K_\Omega m_1 m_2 K_\Omega}. \tag{III.1}$$

□

**Remark** III.7.0.4. *If $K'$ is any compact subgroup admitting an Iwahori factorization with respect to $B$ (as in the above remark), we get a homomorphism of $\mathbb{Z}$-rings*

$$\mathbb{Z}[M^-] \longrightarrow \mathcal{H}_{K'}(\mathbb{Z}).$$

*If, in addition, $M \cap K'$ is normal in $M$ and $M/M \cap K'$ abelian (for example $G_1$-fixators or $G^1$-fixators of bounded subset of $\mathcal{A}_{\text{red}}$ containing an alcove.), then the above construction yields an homomorphism of rings*

$$\mathbb{Z}[M^-(M \cap K')/M \cap K'] \longrightarrow \mathcal{H}_{K'}(\mathbb{Z})$$

*with commutative image.*

**Lemma** III.7.0.4. *If $K_\Omega$ is the compact open subgroup defined in Remark III.7.0.2, then for any $m \in M^-$*

$$[K_\Omega m K_\Omega : K_\Omega] = \delta_B(m)^{-1}.$$

In particular, if $\Omega = \mathfrak{a}$, then we get $q_m = [ImI : I] = \delta_B(m)^{-1}$ for all $m \in M^-$.

*Proof.* Consider the map $i_1 m i_2 \mapsto i_1$ from the set $K_\Omega m K_\Omega$ to $K_\Omega$. This yields a bijection

$$K_\Omega m K_\Omega / K_\Omega \xrightarrow{\sim} K_\Omega / K_\Omega \cap m K_\Omega m^{-1}.$$

Here, $K_\Omega \cap m K_\Omega m^{-1} = K_\Omega \cap K_{\nu(m)(\Omega)}$ is the $K_{\Omega_m^{\mathrm{conv}}} = P(F^{un})^\circ_\Omega \cap G_1$ the $G_1$-fixator of the convex hull of $\Omega_m^{\mathrm{conv}} := \Omega \cup \nu(m)(\Omega)$. Since $m \in M^-$, then $K_\Omega \cap m K_\Omega m^{-1} = U_\Omega^- M_1 m U_\Omega^+ m^{-1}$ and so

$$\begin{aligned} [K_\Omega : K_\Omega \cap m K_\Omega m^{-1}] &= [U_\Omega^+ : m U_\Omega^+ m^{-1}] \\ &= \delta_B^{-1}(m). \end{aligned}$$

□

## III.8 The module $\mathcal{M}_I(\mathbb{Z})$

Following the approach of [HKP10], we define the universal unramified principal series right $\mathcal{H}_I(\mathbb{Z})$-module $\mathcal{M}_I(\mathbb{Z}) = \mathcal{C}_c(M_1 U^+ \backslash G / I, \mathbb{Z})$, this is the set of $\mathbb{Z}$-valued functions supported on finitely many double cosets. The $\mathcal{H}_I(\mathbb{Z})$-module structure of $\mathcal{M}_I(\mathbb{Z})$ comes from the natural right convolution action[14]. There is a natural identification[15] between $\mathcal{R} := \mathbb{Z}[\Lambda_M]$ and the Iwahori–Hecke algebra $\mathcal{H}(M \mathbin{/\!/} M_1, \mathbb{Z})$ that allows us to endow the $\mathcal{H}_I(\mathbb{Z})$-module $\mathcal{M}_I(\mathbb{Z})$ with a left $\mathcal{R}$-action as follows: define for every $\psi \in \mathcal{M}_I(\mathbb{Z})$ and $r \in \mathcal{R}$:

$$r \cdot \psi(g) := \int_M r(a) \psi(a^{-1} g) d\mu_{M_1}(a) \quad (\forall g \in G)$$

here, $d\mu_{M_1}(a)$ is the Haar measure on $M$ giving $M_1$ volume 1. We will see in Lemma III.8.0.2 a more concrete description of this action. It is clear that the actions of $\mathcal{H}_I(\mathbb{Z})$ and $\mathcal{R}$ on $\mathcal{M}_I(\mathbb{Z})$ commute: $\mathcal{M}_I(\mathbb{Z})$ is an $(\mathcal{R}, \mathcal{H}_I(\mathbb{Z}))$-bimodule.

**Remark** III.8.0.1. *Here, we have defined an untwisted action without using the modulus character. The twisted action is defined as follows*

$$r \cdot_{twist} \psi(m) := \int_M \delta_B^{1/2}(a) r(a) \psi(a^{-1} m) d\mu_{M_1}(a).$$

[14] For every $\psi \in \mathcal{M}_I(\mathbb{Z})$ and $h \in \mathcal{H}_I(\mathbb{Z})$, the action is given by the convolution $\psi *_I h$.

[15] This is a consequence of the fact that $M_1$ is the unique parahoric (in particular, an Iwahori) subgroup of M.

In Theorem III.8.0.1, we will generalize [HKP10, Lemma 1.6.1]. For this purpose, we begin by the following two lemmas:

**Lemma** III.8.0.1. *There is a canonical bijection*

$$\widetilde{W} \xrightarrow{\simeq} M_1U^+\backslash G/I \ .$$

*Proof.* Consider the natural map $\widetilde{W} \to M_1U^+\backslash G/I$; $nM_1 \mapsto M_1U^+nI$, it is clearly well defined. The surjectivity follows from the equalities:

$$\begin{aligned}
G &= BK && \text{Proposition III.7.0.3}\\
&= \cup_{w\in W} BwI && \text{Proposition III.7.0.2}\\
&= BNI &&\\
&= M_1U^+NI && \text{Levi factorization of } \mathbf{B}.
\end{aligned}$$

To show injectivity, let $n_1, n_2 \in N$ having same image in $M_1U^+\backslash G/I$;

$$\begin{aligned}
n_2 \in U^+M_1n_1I_{\mathfrak{a}} &= U^+M_1I_{\nu_N(n)(\mathfrak{a})}n_1 && \text{Proposition II.3.4.1 (1)}\\
&= U^+I_{\nu_N(n)(\mathfrak{a})}n_1 && M_1 \subset I_{\nu_N(n)(\mathfrak{a})}\\
&= U^+U_{\nu_N(n)(\mathfrak{a})}M_1n_1 && \text{Proposition II.3.9.2}\\
&= U^+(U^+ \cap I_{\nu_N(n)(\mathfrak{a})})(U^- \cap I_{\nu_N(n)(\mathfrak{a})})M_1n_1\\
&= U^+(U^- \cap I_{\nu_N(n)(\mathfrak{a})})M_1n_1.
\end{aligned}$$

There exists then $m \in M_1$ such that $n_2n_1^{-1}m^{-1} \in U^+(U^- \cap I_{w(\mathfrak{a})})$. But since $U^+U^- \cap N = \{1\}$ [BT65, 5.15], we must have $mn_1 = n_2$, i.e. $M_1n_1 = M_1n_2$. In conclusion, we get a canonical bijective map $\widetilde{W} \to M_1U^+\backslash G/I$. □

**Remark** III.8.0.2. *The preceding proof may be generalized as follows. We continue with the notation of Remark III.7.0.1. We have*

$$\begin{aligned}
G &= BNI\\
&= M_1U^+NK_{\mathcal{F}} && I \subset K_{\mathcal{F}}\\
&= \bigcup_{n\in N/N\cap K_{\mathcal{F}}} M_1U^+nK_{\mathcal{F}}
\end{aligned}$$

*Let $n_1, n_2 \in N$ such that $M_1U^+n_1K_\mathcal{F} = M_1U^+n_2K_\mathcal{F}$. This is equivalent to*

$$\begin{aligned} n_2 \in U^+M_1n_1K_\mathcal{F} &= U^+K_{\nu_N(n)(\mathcal{F})}n_1 && \text{Proposition II.3.4.1 (1)} \\ &= U^+U_{\nu_N(n)(\mathcal{F})}M_1n_1 && \text{Proposition II.3.9.2} \\ &= U^+(U^+ \cap K_{\nu_N(n)(\mathcal{F})})(U^- \cap K_{\nu_N(n)(\mathcal{F})})M_1n_1 \\ &= U^+(U^- \cap K_{\nu_N(n)(\mathcal{F})})M_1n_1. \end{aligned}$$

*Using again $U^+U^- \cap N = \{1\}$ [BT65, 5.15], we see that $n_1n_2^{-1} \in M_1 \subset N \cap K_\mathcal{F}$. Hence $n_1(N \cap K_\mathcal{F}) = n_2(N \cap K_\mathcal{F})$, this implies*

$$G = \bigsqcup_{n \in N/N \cap K_\mathcal{F}} M_1U^+nK_\mathcal{F}. \tag{III.2}$$

*We have defined in Remark III.7.0.1*

$$\widetilde{W}^{K_\mathcal{F}} := N \cap K_\mathcal{F}/M_1 \subset \widetilde{W},$$

*for example: if $\mathcal{F} = \mathfrak{a}$ then $\widetilde{W}^{K_\mathfrak{a}}$ is the trivial subgroup, and if $\mathcal{F} = a_\circ$ the fixed special vertex then $\widetilde{W}^{K_{a_\circ}} = N \cap K/M_1 = W$. Combining the decomposition $G = INK_\mathcal{F}$ and the last isomorphism in [HR08, Remark 9], we get*

$$M \cap I\backslash N/N \cap K_\mathcal{F} \xrightarrow{\simeq} I\backslash G/K_\mathcal{F} \xrightarrow{\simeq} \widetilde{W}/\widetilde{W}^{K_\mathcal{F}} \ ,$$

*but $M_1 = N \cap I$ (Lemma III.7.0.1) is normal in $N$ and $M_1 \subset N \cap K_\mathcal{F}$, thus we actually have a bijection $M_1\backslash N/N \cap K_\mathcal{F} \simeq N/N \cap K_\mathcal{F}$. We may then rewrite (III.2) as follows $G = \bigsqcup_{w \in \widetilde{W}/\widetilde{W}^{K_\mathcal{F}}} M_1U^+wK_\mathcal{F}$, this gives a bijection generalizing the above Lemma*

$$M_1U^+\backslash G/K_\mathcal{F} \xrightarrow{\simeq} \widetilde{W}/\widetilde{W}^{K_\mathcal{F}} \tag{III.3}$$

An immediate consequence of Lemma III.8.0.1 is:

**Corollary** III.8.0.1. *The family*

$$\{v_w := \mathbf{1}_{M_1U^+wI} \colon w \in \widetilde{W}\},$$

*forms a $\mathbb{Z}$-basis for the $\mathbb{Z}$-module $\mathcal{M}_I(\mathbb{Z})$.*

**Lemma** III.8.0.2. *The action of $\mathcal{R} = \mathbb{Z}[\Lambda_M]$ on $\mathcal{M}_I(\mathbb{Z})$ is induced by the action of $\Lambda_M = M/M_1$ on $M_1U^+\backslash G/I$, so for any $w \in \widetilde{W}$ and $m \in \Lambda_M$ we have*

$$m \cdot v_w = v_{mw}.$$

*Proof.* Let $w \in \widetilde{W}$ and $r = mM_1 \in \mathcal{R}$ for some $m \in M$. Recall that $\mathcal{R}$ is identified with the Iwahori–Hecke algebra for $M$ (see footnote 15 on page 83); $r = mM_1 \leftrightarrow r = \mathbf{1}_{mM_1}$.

We have defined

$$r \cdot v_w(b) = \int_M r(m')v_w(m'^{-1}b)d\mu_{M_1}(m'),$$

The integral is non-zero only if $b \in mM_1U^+wI = M_1U^+mwI$ ($m \in M$ normalizes $U^+$), hence $r \cdot v_w = s\, v_{mw}$ where $s$ is the scalar $r \cdot v_w(mw) \in \mathbb{Z}$:

$$\begin{aligned} r \cdot v_w(mw) &= \int_M r(m')v_w(w)d\mu_{M_1}(m') \\ &= \int_M \mathbf{1}_{mM_1}(m')d\mu_{M_1}(m') \\ &= |mM_1| = 1. \quad \square \end{aligned}$$

**Proposition** III.8.0.1. *The action of $\mathcal{H}_I(\mathbb{Z})$ on $\mathcal{M}_I(\mathbb{Z})$ is described by the following rules: for every $w \in W$ and $m \in \Lambda_M$, we have*

1. $v_1 *_I i_w = v_w$,
2. $v_m *_I i_w = v_{mw}$,
3. $v_1 *_I i_m = v_m$ *if* $m \in \Lambda_M^-$.

*Proof.* 1. Let $w \in W$. We abuse notation and write also $w$ for any representative in $N \cap K$ of the class $w \in W$. If the quantity $v_1 *_I i_w(b) = \int_{M_1U^+I} i_w(a^{-1}b)d\mu_I(a)$ is non-zero for some $b$, then there exists $a \in M_1U^+I$ such that $b \in aIwI$, hence $b \in M_1U^+I \cdot IwI$. By corollary III.7.0.2, for each $w \in W$, we have $wI^-w^{-1} \subset I$. Hence,

$$b \in M_1U^+I \cdot IwI = M_1U^+I^+wI = M_1U^+wI.$$

Thus, $v_1 *_I i_w = s\, v_w$, for some scalar $s \in \mathbb{Z}$. It suffices then to compute $v_1 *_I i_w(w) = s$. The only elements of $G$ which contribute to the integral

$$\int_{M_1U^+I} i_w(a^{-1}w)d\mu_I(a),$$

are those in $M_1U^+I \cap wIw^{-1}I$. This set is equal to $I$, indeed:

$$I \subset M_1U^+I \cap wIw^{-1}I \subset M_1U^+I \cap K = M_1(U^+ \cap K)I = I,$$

where, we have used $w \in K$ for the second inclusion, and the last equality is a consequence of $U^+ \cap K = I^+$ (see proof of Proposition III.7.0.2). In conclusion, we have proved that

$$s = \int_{(M_1U^+I \cap wIw^{-1}I)} d\mu_I(a) = 1.$$

2. To prove the third equality we use (1) and Lemma III.8.0.2 above. Indeed

$$\begin{aligned} v_m *_I i_w &= (m \cdot v_1) *_I i_w \\ &= m \cdot (v_1 *_I i_w) \\ &= m \cdot v_w = v_{mw}. \end{aligned}$$

3. Let $m \in M^-$. The value of $v_1 *_I i_m(b) = \int_{M_1U^+I} i_m(a^{-1}b)d\mu_I(a)$ is non zero only if $b = aa^{-1}b \in M_1U^+I \cdot ImI$. Using the lemma above and the Iwahori factorization of $I$ we obtain

$$\begin{aligned} M_1U^+I \cdot ImI &= M_1U^+ImI \\ &= M_1U^+I^+I^-mI && (I = M_1I^+I^-) \\ &= M_1U^+I^-mI \\ &= M_1U^+mm^{-1}I^-mI \\ &= M_1U^+mI && \text{Lemma III.7.0.3} \end{aligned}$$

Thus, $v_1 *_I i_m = s\, v_m$ with $s = (v_1 *_I i_m)(m) = \mathrm{vol}(M_1U^+I \cap mIm^{-1}I)$. We claim that

$$M_1U^+I \cap mIm^{-1}I = I.$$

Recall that $M_1$ normalizes $U^+$, so

$$\begin{aligned} M_1U^+I \cap mIm^{-1}I &= U^+I \cap mIm^{-1}I \\ &= U^+I \cap mI^-m^{-1}I && \text{Lemma III.7.0.3} \\ &= (U^+ \cap mI^-m^{-1}I)I \end{aligned}$$

Using again Lemma III.7.0.3 and the Iwahori factorization of $I$, the intersection between the parentheses is:

$$U^+ \cap mI^-m^{-1}I = U^+ \cap M_1(mI^-m^{-1})I^+.$$

Let $u \in U^+ \cap M_1(mI^-m^{-1})I^+$, so $u = m_1i_1i_2$ for some $m_1 \in M_1, i_1 \in mI^-m^{-1} \subset U^-, i_2 \in I^+$, and

$$m_1 = (ui_2^{-1})i_1^{-1} \in U^+U^- \cap N.$$

Now, because $U^+U^- \cap N = \{1\}$ [BT65, 5.15], we see that[16] $m_1 = 1$, $u = i_2$ and $i_1 = 1$. This shows that $U^+ \cap M_1mI^-m^{-1}I^{-1} = I^+$ and consequently the claim

$$M_1U^+I \cap mIm^{-1}I = I^+I = I.$$

[16] Also since, unipotent radicals of opposite parabolics have trivial intersection.

In conclusion, we have shown that

$$v_1 *_I i_m = v_1 *_I i_m(m)\, v_m = \mathrm{vol}(U^+I \cap mIm^{-1}I)v_m = v_m. \quad \square$$

The following proposition gives a precise description of the $\mathcal{R}$-module $\mathcal{M}_I(\mathbb{Z})$:

**Proposition** III.8.0.2. *The $\mathcal{R}$-module $\mathcal{M}_I(\mathbb{Z})$ is a free module of rank $|W|$, the size of the finite Weyl group, with canonical basis $\{v_w,\ w \in W\}$.*

*Proof.* Recall that by Corollary III.8.0.1 the family $\{v_x\}$ for $x \in \widetilde{W} = \Lambda_M \rtimes W$ is a $\mathbb{Z}$-basis for $\mathcal{M}_I(\mathbb{Z})$. We can write each $x$ as $x = m_x w_x \in \widetilde{W}$ for a unique $m_x \in \Lambda_M$ and unique $w_x \in W$. By Lemma III.8.0.2 above, we have

$$v_x = m_x \cdot v_{w_x}.$$

This proves that $\{v_w\}_{w\in W}$ is a generating and linearly independent set of the $\mathcal{R}$-module $\mathcal{M}_I(\mathbb{Z})$ (Lemma III.8.0.1). $\square$

We would like to describe the structure of $\mathcal{M}_I(\mathbb{Z})$, this time as a $\mathcal{H}_I(\mathbb{Z})$-module. However, a satisfactory result can only be obtained after enlarging the coefficients ring. Set $R := \mathbb{Z}[q^{\pm 1}]$.

**Theorem** III.8.0.1. *The following homomorphism of right $\mathcal{H}_I(R)$-modules*

$$\begin{aligned} \mathcal{H}_I(R) &\longrightarrow \mathcal{M}_I(R) \\ h &\longmapsto v_1 *_I h \end{aligned}$$

*is an isomorphism.*

*Proof.* It suffices to show that the map $h \mapsto v_1 *_I h$ is "upper triangular with respect to the Chevalley–Bruhat order, with invertible diagonals". Recall that $\mathcal{H}_I(\mathbb{Z})$ (resp. $\mathcal{M}_I(\mathbb{Z})$) admits the $\mathbb{Z}$-basis $i_y$ (resp. $v_x$) for $x, y \in \widetilde{W}$. We have

$$v_1 *_I i_y(b) = \int_G v_1(ba) i_y(a^{-1}) d\mu_I(a).$$

We can write $v_1 *_I i_y = \sum_{x\in\widetilde{W}} c_{x,y} v_x$ with $c_{x,y} \in \mathbb{Z}$. If $x \in \widetilde{W}$ such that $c_{x,y} \neq 0$, then $v_1 *_I i_y(x) \neq 0$. Since $v_1 *_I i_y$ can be non-zero only on the set $M_1U^+IyI$, so $x \in M_1U^+IyI$. This implies that $M_1U^+xI \cap IyI \neq \emptyset$.

**Lemma** III.8.0.3. *Let $x, y \in \widetilde{W}$. If $M_1U^+xI \cap IyI \neq \emptyset$ then $x \leq y$ in the Chevalley–Bruhat order.*

*Proof.* The proof resembles the proof of the claim in [HKP10, Lemma 1.6.1]. Let $m \in M_1$ and $u \in U^+$ such that $mux = iyi' \in IyI$ for some $i, i' \in I$. Choose a cocharacter $\mu \in X_*(\mathbf{S})$ dominant enough to ensure that $\varpi^\mu u \varpi^{-\mu} \in I$[17]. Therefore,

$$I\varpi^\mu xI = Im(\varpi^\mu u\varpi^{-\mu})\varpi^\mu xI = I\varpi^\mu iyi'I \subset I\varpi^\mu IyI.$$

Now, we use Corollary III.3.0.1 and deduce that

$$I\varpi^\mu xI \subset \bigsqcup_{z\leq y} I\varpi^\mu zI,$$

which is equivalent to the desired inequality $x \leq y$. □

Using Lemma III.8.0.3, we obtain the upper triangularity:

$$v_1 *_I i_y = \sum_{x\in\widetilde{W}:\, x\leq y} c_{x,y}v_x.$$

For all $y \in \widetilde{W}$, the corresponding diagonal coefficient is:

$$c_{y,y} = v_1 *_I i_y(y) = \int_G v_1(ya)i_y(a^{-1})d\mu_I(a) = |y^{-1}U^+I \cap Iy^{-1}I|.$$

Note that $c_{y,y} \geq |y^{-1}I| = 1$ and is a power of $q$ by Lemma III.8.0.4 below. Hence, $c_{y,y} \in R^\times$ for all $y \in \widetilde{W}$. This proves that the matrix $(c_{x,y})_{x,y\in\widetilde{W}}$ is indeed invertible once we extend the coefficients from $\mathbb{Z}$ to $R$. □

**Lemma** III.8.0.4. *For any $n \in N$ the volume $|nU^+I \cap InI|$ is a power of $q$.*

*Proof.* Consider the map $i_1ai_2 \mapsto i_1$ from the set $nU^+I \cap InI$ to $I$. Here, if $i_1ni_2$ lies in $nU^+I \cap InI$ then $i_1 \in nU^+In^{-1} \cap I$, this yields a bijection (since $nU^+In^{-1} \cap nIn^{-1} \cap I = nIn^{-1} \cap I$)

$$nU^+I \cap InI/I \xrightarrow{\sim} nU^+In^{-1} \cap I/nIn^{-1} \cap I.$$

We denote by $w_n \in W$ the image of $n$. Set

$$\Phi_n^\pm := \{w_n(\alpha) \in \Phi \colon \alpha \in \Phi^\pm\}.$$

By (4) and (5) of Proposition II.3.4.1 we have for any fixed ordering $\prec_\pm$ of $\Phi_n^\pm$:

$$I = \left(\prod_{\Phi_{\mathrm{red}}\cap\Phi_n^+}^{\prec_+} U_{\alpha+f_{\mathfrak{a}}(\alpha)}\right)\cdot M_1 \cdot \left(\prod_{\Phi_{\mathrm{red}}\cap\Phi_n^-}^{\prec_-} U_{\alpha+f_{\mathfrak{a}}(\alpha)}\right)$$

[17] By abuse of notation, we write $\varpi^\mu$ for $\varpi(\mu)$.

recall that

$$f_{\mathfrak{a}}(\alpha) = \begin{cases} 0 \text{ if } \alpha \in \Phi^+ \\ n_\alpha^{-1} \text{ if } \alpha \in \Phi^-. \end{cases}$$

We also have

$$nU^+M_1I^-n^{-1} = \left( \prod_{\Phi_{\mathrm{red}} \cap \Phi_n^+}^{\prec_+} \mathbf{U}_\alpha(F) \right) \cdot M_1 \cdot \left( \prod_{\Phi_{\mathrm{red}} \cap \Phi_n^+}^{\prec_-} U_{\beta_\alpha(n)} \right),$$

where, $\beta_\alpha(n)\colon b \mapsto w_n(\alpha)(b - \nu_N(n))(a_\circ)) + n_\alpha^{-1}$, thus

$$\beta_\alpha(n) = w_n(\alpha) + r_\alpha(n), \quad r_\alpha(n) = w_n(\alpha)(a_\circ - \nu_N(n))(a_\circ) + n_\alpha^{-1} \in \mathbb{R}.$$

The fact that $\mathbf{U}^+_{\Phi_n}(F)\mathbf{U}^-_{\Phi_n}(F) \cap N = \{1\}$ and $\mathbf{U}^+_{\Phi_n}(F) \cap \mathbf{U}^-_{\Phi_n}(F) = \{1\}$ shows

$$nU^+M_1I^-n^{-1} \cap I = \left( \prod_{\Phi_{\mathrm{red}} \cap \Phi_n^+}^{\prec_+} U_{\alpha + f_{\mathfrak{a}}(\alpha)} \right) \cdot M_1 \cdot \left( \prod_{\Phi_{\mathrm{red}} \cap \Phi_n^-}^{\prec_-} U_{\alpha + \max(f_{\mathfrak{a}}(\alpha), r_{w_n^{-1}(\alpha)})} \right).$$

Similarly, we can show that

$$nM_1In^{-1} \cap I = \left( \prod_{\Phi_{\mathrm{red}} \cap \Phi_n^+}^{\prec_+} U_{\alpha + \max(f_{\mathfrak{a}}(\alpha), r_{w_n^{-1}(\alpha)})} \right) \cdot M_1 \cdot \left( \prod_{\Phi_{\mathrm{red}} \cap \Phi_n^-}^{\prec_-} U_{\alpha + \max(f_{\mathfrak{a}}(\alpha), r_{w_n^{-1}(\alpha)})} \right).$$

Therefore,

$$nU^+M_1I^-n^{-1} \cap I / nM_1I^-n^{-1} \cap I \simeq \prod_{\Phi_{\mathrm{red}} \cap \Phi_n^+}^{\prec_+} U_{\alpha + f_{\mathfrak{a}}(\alpha)} / U_{\alpha + \max(f_{\mathfrak{a}}(\alpha), r_{w_n^{-1}(\alpha)})}$$

The size of each quotient in the right product is a power of $q$. □

The above theorem, implies immediately the following

**Corollary** III.8.0.2. *The right $\mathcal{H}_I(R)$-module $\mathcal{M}_I(R)$ is free and of rank 1, with canonical generator $v_1$. This yields a canonical isomorphism*

$$\mathcal{H}_I(R) \simeq End_{\mathcal{H}_I(R)}(\mathcal{M}_I(R)),$$

*sending $h \in \mathcal{H}_I(R)$ to the endomorphism $v_1 *_I h' \mapsto v_1 *_I h *_I h'$.*

## III.9 Decomposition of $\mathcal{H}_I(R)$ and the untwisted Bernstein map

We begin this section by considering two sub-algebras of the Iwahori–Hecke algebra $\mathcal{H}_I(\mathbb{Z})$:

- The finite Hecke algebra:

  $$\mathcal{H}_I(\mathbb{Z})^0 := \mathcal{C}_c(K /\!\!/ I, \mathbb{Z}).$$

  A $\mathbb{Z}$-basis for $\mathcal{H}_I(\mathbb{Z})^0$ is given by $\{i_w\}_{w\in W}$ (Proposition III.7.0.1).

- The $(\mathcal{R}, \mathcal{H}_I(\mathbb{Z}))$-bimodule structure on $\mathcal{M}_I(\mathbb{Z})$ induces a homomorphism of algebras [18]

  $$\mathcal{R} \to \mathrm{End}_{\mathcal{H}_I(\mathbb{Z})}(\mathcal{M}_I(\mathbb{Z})),$$

  which is actually an embedding since $\mathcal{M}_I(\mathbb{Z})$ is free over $\mathcal{R}$.Composing the above embedding with the canonical isomorphism[19] $\mathrm{End}_{\mathcal{H}_I(R)}(\mathcal{M}_I(R)) \simeq \mathcal{H}_I(R)$ (Theorem III.8.0.1), we obtain an embedding of algebras:

  $$\begin{aligned} \dot{\Theta}_{\mathrm{Bern}} \colon \mathcal{R} &\hookrightarrow \mathcal{H}_I(R) \\ m &\mapsto \dot{\Theta}_m, \end{aligned}$$

  characterized by the property: $m \cdot v_1 = v_1 *_I \dot{\Theta}_m$, for every $m \in \mathcal{R}$.

**Remark** III.9.0.1. *Elements of $\Lambda_M \subset \mathcal{R}^\times$ act invertibly on $\mathcal{M}_I(R)$, i.e.*

$$\dot{\Theta}_{Bern}(\Lambda_M) \subset \mathcal{H}_I(R)^\times.$$

*Therefore, for every $m \in \Lambda_M^-$, the Iwahori–Hecke operator $i_m \in \mathcal{H}_I(R)$ is invertible with inverse $\dot{\Theta}_{-m}$, since by (3) Proposition III.8.0.1, we have $i_m = \dot{\Theta}_m$ for $m \in \Lambda_M^-$ (See also Remark III.6.0.1).*

In our setting, it is easy to obtain a statement similar to [HKP10, Lemma 1.7.1]:

**Lemma** III.9.0.1. *The homomorphism of $\mathcal{R} \otimes_{\mathbb{Z}} R$-modules*

$$\begin{aligned} \mathcal{R} \otimes_{\mathbb{Z}} \mathcal{H}_I^0(R) = (\mathcal{R} \otimes_{\mathbb{Z}} \mathcal{H}_I^0(\mathbb{Z})) \otimes_{\mathbb{Z}} R &\longrightarrow \mathcal{H}_I(R) \\ m \otimes h &\longmapsto \dot{\Theta}_m *_I h, \end{aligned}$$

*is an isomorphism. Composing this homomorphism with $h \mapsto v_1 *_I h$ yields the isomorphism of $\mathcal{R} \otimes_{\mathbb{Z}} R$-modules*

$$\mathcal{R} \otimes_{\mathbb{Z}} \mathcal{H}_I^0(R) \longrightarrow \mathcal{M}_I(R),$$

*given by $m \otimes i_w \longmapsto v_{mw}$, for $w \in W$ and $m \in \Lambda_M$.*

[18] In other words, since the action of $\mathcal{R}$ on $\mathcal{M}_I(\mathbb{Z})$ commutes with the action of $\mathcal{H}_I(\mathbb{Z})$, elements of $\mathcal{R}$ may be viewed as endomorphisms of $\mathcal{M}_I(\mathbb{Z})$.

[19] More precisely, we mean the composition of the canonical embeddings

$$\mathcal{R} \hookrightarrow \mathrm{End}_{\mathcal{H}_I(\mathbb{Z})}(\mathcal{M}_I(\mathbb{Z})) \hookrightarrow \mathrm{End}_{\mathcal{H}_I(R)}(\mathcal{M}_I(R)) \simeq \mathcal{H}_I(R).$$

*Proof.* Let us first show that the composition of the maps

$$\mathcal{R} \otimes_{\mathbb{Z}} \mathcal{H}_I^0(R) \longrightarrow \mathcal{H}_I(R) \longrightarrow \mathcal{M}_I(R),$$

yields an isomorphism of $\mathcal{R} \otimes_{\mathbb{Z}} R$-modules. For every $m \in \Lambda_M$ and $w \in W$, the image of $m \otimes i_w$ is $v_1 *_I \dot{\Theta}_m *_I i_w$:

$$\begin{aligned} v_1 *_I \dot{\Theta}_m *_I i_w &= m \cdot v_1 *_I i_w && \text{Definition of } \dot{\Theta}_m \\ &= m \cdot v_w && \text{(1) Proposition III.8.0.1} \\ &= v_{mw} && \text{Lemma III.8.0.2.} \end{aligned}$$

This formula shows in particular that the induced isomorphism $\mathcal{R} \otimes_{\mathbb{Z}} \mathcal{H}_I^0(R) \to \mathcal{M}_I(R)$ is defined over $\mathbb{Z}$. Now, since $\mathcal{H}_I(R) \to \mathcal{M}_I(R)$ is an isomorphism by Theorem III.8.0.1, the first map $\mathcal{R} \otimes_{\mathbb{C}} \mathcal{H}_I^0(R) \to \mathcal{H}_I(R)$ must also be an isomorphism. □

A direct application of Lemma III.9.0.1 gives an explicit $\mathcal{R} \otimes_{\mathbb{Z}} R$-basis for the Iwahori–Hecke algebra:

**Corollary** III.9.0.1. *The Iwahori–Hecke algebra $\mathcal{H}_I(R)$ is a free left $\mathcal{R} \otimes_{\mathbb{Z}} R$-module* [20], *with canonical basis $\{i_w : w \in W\}$. The sets $\{\dot{\Theta}_m *_I i_w : m \in \Lambda_M, w \in W\}$ is an $R$-basis for $\mathcal{H}_I(R)$.*

In the following proposition, we give an explicit formula for $\dot{\Theta}_m$, for all $m \in \Lambda_M$.

**Proposition** III.9.0.1. *Let $m \in \Lambda_M$, then*

$$\dot{\Theta}_m = i_{m_1} *_I (i_{m_2})^{-1} = (i_{m_2})^{-1} *_I i_{m_1},$$

*for any $m_1, m_2 \in \Lambda_M^-$ satisfying $m = m_1 - m_2$.*

*Proof.* Let $m = \Lambda_M$. By Proposition III.4.0.1, there exists $m_1, m_2 \in \Lambda_M^-$ such that $m = m_1 - m_2$. Therefore,

$$\begin{aligned} \dot{\Theta}_m &= \dot{\Theta}_{\text{Bern}}(m) \\ &= \dot{\Theta}_{\text{Bern}}(m_1 - m_2) \\ &= \dot{\Theta}_{\text{Bern}}(m_1) *_I \dot{\Theta}_{\text{Bern}}(-m_2) \\ &= \dot{\Theta}_{m_1} *_I \dot{\Theta}_{-m_2} \\ &= i_{m_1} *_I (i_{m_2})^{-1} = (i_{m_2})^{-1} *_I i_{m_1}. \quad \square \end{aligned}$$

[20] We implicitly identify $\mathcal{R}$ with its image $\dot{\Theta}_{\text{Bern}}(\mathcal{R}) \subset \mathcal{H}_I(R)$.

**Remark** III.9.0.2. *The Iwahori–Hecke operators $\dot{\Theta}_m$ generalizes the element denoted by $\widetilde{T}$ in Lusztig's [Lus83, §7] (up to the factor $\delta_B(m)^{1/2}$), which does not appear here because of our untwisted definition of the action of $\mathcal{R}$ on $\mathcal{M}_I(R)$. The twisted version of the map $\dot{\Theta}_{\mathrm{Bern}}$ was attributed by Lusztig (for split reductive groups) to Bernstein [Lus83, §7], this is the reason we use the subscript* Bern.

**Remark** III.9.0.3. *For $m, m_1$ and $m_2$ as in Proposition III.9.0.1, we have an alternative expression for $\dot{\Theta}_{\mathrm{Bern}}(m)$:*

$$\begin{aligned}\dot{\Theta}_{\mathrm{Bern}}(m) &= \frac{1}{q_{m_2}} i_{m_1} *_I i^*_{m_2} && \text{Remark III.6.0.1}\\ &= \delta_B(m_2) i_{m_1} *_I i^*_{m_2} && \text{Lemma III.7.0.4.} \ \square\end{aligned}$$

**Remark** III.9.0.4. *Observe that the subalgebra of $\mathcal{H}_I(R)$ generated by $\{\dot{\Theta}_m : m \in \Lambda_M\}$ is commutative by definition ($= \dot{\Theta}_{\mathrm{Bern}}(\mathcal{R})$).*

## III.10 Untwisted Satake isomorphism

Motivated by arithmetic problems for Shimura varieties, Haines and Rostami [HR10] established the following Satake isomorphism type $\mathcal{H}_K(\mathbb{C}) \simeq \mathcal{R}^W \otimes_{\mathbb{Z}} \mathbb{C}$ (Recall that $K$ is a special maximal parahoric subgroup of $G$). In this section, we will construct an untwisted version of this homomorphism and prove an isomorphism

$$\mathcal{H}_K(R) \simeq (\mathcal{R} \otimes_{\mathbb{Z}} R)^{(W,\bullet)},$$

where, $(W, \bullet)$ denotes the Weyl invariant for an "untwisted" action of $W$ denoted $\bullet$. From this isomorphism we will get an isomorphism

$$\mathcal{H}_K(\mathbb{Z}) \simeq \mathcal{R}^{(W,\bullet)},$$

here $\mathcal{R}^{(W,\bullet)}$ denotes the $\mathbb{Z}$-submdule of $\mathcal{R}$ of Weyl-invariant for the $\bullet$-action.

We begin by the right $\mathcal{H}_K(\mathbb{Z})$-module $\mathcal{M}_K(\mathbb{Z}) := \mathcal{C}_c(M_1U^+\backslash G/K, \mathbb{Z})$, on which $\mathcal{H}_K(\mathbb{Z})$ acts from the right by convolution with respect to the normalized measure $\mu_K$ giving $K$ volume 1.

**Remark** III.10.0.1. *We can naturally identify $\mathcal{H}_K(\mathbb{Q})$ with the two-sided ideal $e_K *_I \mathcal{H}_I(\mathbb{Q}) *_I e_K \subset \mathcal{H}_I(\mathbb{Q})$, where $e_K$ is the idempotent $|K : I|^{-1}\mathbf{1}_K$ (see Lemma III.5.0.3). In a similar way, $\mathcal{M}_K(\mathbb{Q})$ can also be naturally identified with $\mathcal{M}_I(\mathbb{Q}) *_I e_K \subset \mathcal{M}_I(\mathbb{Q})$, and the right action of $\mathcal{H}_K(\mathbb{Q})$ will correspond to the right action of $e_K *_I \mathcal{H}_I(\mathbb{Q}) *_I e_K$.*

We define the action of $\mathcal{R}$ on $\mathcal{M}_K(\mathbb{Z})$ in a same way we have defined its action on $\mathcal{M}_I(\mathbb{Z})$. This way, $\mathcal{M}_K(\mathbb{Z})$ inherits a structure of an $(\mathcal{R}, \mathcal{H}_K(\mathbb{Z}))$-bi-module.

**Lemma** III.10.0.1. *The $\mathcal{H}_K(\mathbb{Z})$-module $\mathcal{M}_K$ is free of rank 1 as $\mathcal{R}$-module (with canonical basis element the spherical vector $v_{1,K} := \mathbf{1}_{M_1 U K}$).*

*Proof.* Using the Iwasawa decomposition $G = BK = MU^+K$, one shows that $M_1U^+\backslash G/K \simeq M/M_1 = \Lambda_M$. We can also use Remark III.8.0.2, indeed we have the bijections

$$\begin{aligned} M_1U^+\backslash G/K &\simeq \widetilde{W}/\widetilde{W}^K && \text{(III.3)} \\ &\simeq (\Lambda_M \rtimes W)/\widetilde{W}^K && \text{Lemma III.2.0.2} \\ &\simeq \Lambda_M \rtimes W/W && \widetilde{W}^K = W \\ &\simeq \Lambda_M. \end{aligned}$$

This shows that $\mathcal{M}_K(\mathbb{Z})$ admits the following $\mathbb{Z}$-basis $\{v_{m,K} := \mathbf{1}_{M_1U^+mK} : m \in \Lambda_M\}$. In addition, using Lemma III.8.0.2, one shows that $m \cdot v_{1,K} = v_{m,K}$ for any $m \in \Lambda_M$, which ends the proof. $\square$

**Remark** III.10.0.2. *The proceeding proof may be applied more generally. We continue with the notation of Remark III.8.0.2. We have*

$$\begin{aligned} M_1U^+\backslash G/K_{\mathcal{F}} &\simeq \widetilde{W}/\widetilde{W}^{K_{\mathcal{F}}} && \textit{by (III.3)} \\ &\simeq \Lambda_M \rtimes W/\widetilde{W}^{K_{\mathcal{F}}} && \textit{Lemma III.2.0.2} \\ &\simeq \Lambda_M \times (W/W_{\mathcal{F}}), \end{aligned}$$

*where $W_{\mathcal{F}} = \pi_W(\widetilde{W}^{K_{\mathcal{F}}})$ and $\pi_W$ is the natural projection $\widetilde{W} \twoheadrightarrow W$ (see §III.2). This shows that $\mathcal{M}_{K_{\mathcal{F}}}(\mathbb{Z})$ admits the following $\mathbb{Z}$-basis*

$$\{v_{mw,K_{\mathcal{F}}} := \mathbf{1}_{M_1U^+mwK_{\mathcal{F}}} : m \in \Lambda_M, w \in W/W_{\mathcal{F}}\}.$$

*Moreover, we define a left action of $\mathcal{R}$ on $\mathcal{M}_{K_{\mathcal{F}}}(\mathbb{Z})$ by convolution in a similar way to the one introduced in the beginning §III.8. For this action, the proof of Lemma III.8.0.2 shows, mutatis mutandis, that for $m, m' \in \Lambda_M$ and $w \in \widetilde{W}$ we have*

$$m' \cdot v_{mw,K_{\mathcal{F}}} = v_{m'mw,K_{\mathcal{F}}}.$$

*Using this formula, we see that the $\mathcal{R}$-module $\mathcal{M}_{K_{\mathcal{F}}}(\mathbb{Z})$ is free of rank $|W/W_{\mathcal{F}}|$, with a canonical $\mathcal{R}$-basis given by*

$$\{v_{w,K_{\mathcal{F}}} := \mathbf{1}_{M_1U^+wK_{\mathcal{F}}} : w \in W/W_{\mathcal{F}}\}.$$

*This generalization of Lemma III.10.0.1 gives a common generalization of two previous* *results: freeness of $\mathcal{M}_{K_{\mathcal{F}}}(\mathbb{Z})$ over $\mathcal{R}$ with rank size of $W$ (resp. 1) for $\mathcal{F}$ and alcove (resp. a special point). This generalization also gives an alternative proof of the main theorem of [Lan02, Theorem 1.1], giving the dimension of the space of $K_{\mathcal{F}}$-fixed vectors of an unramified principal series representation of $G$.*

By the previous lemma, we obtain a $\mathbb{Z}$-algebra homomorphism $\mathcal{H}_K(\mathbb{Z}) \to \mathcal{R}$, which we denote by $\dot{\mathcal{S}}_M^G$. It is called the Satake transform and is characterized by

$$v_{1,K} *_K h = \dot{\mathcal{S}}_M^G(h) \cdot v_{1,K},$$

for all $h \in \mathcal{H}_K(\mathbb{Z})$. This is actually an embedding of $\mathbb{Z}$-algebras. Indeed, let $h \in \mathcal{H}_K(\mathbb{Z})$ such that $\dot{\mathcal{S}}_M^G(h) = 0$, hence

$$v_1 *_I h = v_1 *_I e_K *_I h = [K : I] v_{1,K} *_K h \in \mathcal{M}_I(R),$$

but since $\mathcal{M}_I(R)$ is a free $\mathcal{H}_K(R)$-module, then $h = 0$. This shows, in particular, that $\mathcal{H}_K(\mathbb{Z})$ is a commutative.

**Lemma** III.10.0.2. *Let $h$ be any function in $\mathcal{H}_K(\mathbb{Z})$, then its Satake untwisted transform is explicitly given by*

$$\dot{\mathcal{S}}_M^G(h) \colon m \mapsto \int_{U^+} h(um) d\mu_{U^+}(u),$$

*where $d\mu_{U^+}$ is the Haar measure on $U^+$ giving 1 on $U^+ \cap K$.*

*Proof.* Let us evaluate $v_{1,K} *_K h = \dot{\mathcal{S}}_M^G(h) \cdot v_{1,K}$ on both sides on $m \in M$: On the one hand, we have

$$\begin{aligned}
(\dot{\mathcal{S}}_M^G(h) \cdot v_{1,K})(m) &= \int_M \dot{\mathcal{S}}_M^G(h)(a) v_{1,K}(a^{-1}m) d\mu_{M_1}(a) \\
&= \int_M \dot{\mathcal{S}}_M^G(h)(a) \mathbf{1}_{M_1 U^+ K \cap M}(a^{-1}m) d\mu_{M_1}(a) \\
&= \int_M \dot{\mathcal{S}}_M^G(h)(a) \mathbf{1}_{M_1}(a^{-1}m) d\mu_{M_1}(a) \\
&= \int_{mM_1} \dot{\mathcal{S}}_M^G(h)(a) d\mu_{M_1}(a) \\
&= \dot{\mathcal{S}}_M^G(h)(m).
\end{aligned}$$

On the second hand we have (For all integration formulas used in the sequel we refer to

[Car79, §4.1])

$$\begin{aligned}
(v_{1,K} *_K h)(m) &= \int_G v_{1,K}(g)h(g^{-1}m)d\mu_K(g) \\
&= \int_B \int_K v_{1,K}(bk)h(k^{-1}b^{-1}m)d\mu_B(b)\,dk \\
&= \int_B v_{1,K}(b)h(b^{-1}m)d\mu_B(b) \qquad (K \text{ invariance}) \\
&= \int_M \int_{U^+} v_{1,K}(au)h(u^{-1}a^{-1}m)d\mu_M(a)d\mu_{U^+}(u) \\
&= \int_{U^+} h(u^{-1}m)d\mu_{U^+}(u) \qquad (v_{1,K}(au) \neq 0 \Rightarrow a \in M_1) \\
&= \int_{U^+} h(um)d\mu_{U^+}(u) \qquad (U^+ \text{ is unimodular})
\end{aligned}$$

Where, $d\mu_B$ (respectively $d\mu_{U^+}$ and $dk$) is the Haar left invariant measure on $B$ (Resp. $U^+$ and $K$) giving volume 1 to $B \cap K$ (respectively $U^+ \cap K$ and $K$). □

**Lemma** III.10.0.3. *Let $h$ be any function in $\mathcal{H}_K(\mathbb{Z})$ and $m \in M$ such that*[21]

$$\Delta(m) := \left|\det(\mathrm{Ad}_{\mathrm{Lie}(U^+)}(m) - \mathrm{Id}_{\mathrm{Lie}(U^+)})\right|_F \neq 0.$$

*Then*

$$\int_{U^+} h(um)d\mu_{U^+}(u) = \Delta(m) \int_{G/S} h(gmg^{-1}) \frac{d\mu_K(g)}{ds}$$

*where, the Haar measure $ds$ on $S = \mathbf{S}(F)$ is normalized by $\int_{M/S} \frac{d\mu_{M_1}}{ds} = 1$.*

*Proof.* See [Car79, Lemma 4.1]. □

**Remark** III.10.0.3. *As opposed to [HR10, Remark 10.1.1], the untwisted Satake transform defined here, is dependent on the choice of the minimal parabolic $\mathbf{B}$ which contains $\mathbf{M}$ as a Levi factor. The reason for this dependence is the absence of the factor due to the modulus function, i.e. in the twisted version, we would have had the factor $D(m) = \Delta(m)\delta_B(m)^{-1/2}$ which verifies $D(nmn^{-1}) = D(m)$ for all $m$ in $M$ and $n$ in $N$.*

**Definition** III.10.0.1 (Dot-action). *We define a twisted action of $W$ on $\mathcal{H}(M /\!/ M_1, R)$, by*

$$w \bullet r := m \mapsto \delta_B(m)^{1/2}\delta_B(n_w^{-1}mn_w)^{-1/2} r(n_w^{-1}mn_w),$$

*for every $w \in W$ represented by $n_w \in N \cap K$, and any $r \in \mathcal{H}(M /\!/ M_1, R)$. In particular, for every $m \in M$,*

$$w \bullet \mathbf{1}_{mM_1} = \delta_B(w(m))^{1/2}\delta_B(m)^{-1/2}\mathbf{1}_{w(mM_1)}.$$

[21] The function $m \mapsto \det(\mathrm{Ad}_{\mathrm{Lie}(U^+)}(m) - \mathrm{Id}_{\mathrm{Lie}(U^+)})$ is polynomial and nonzero. The set of regular elements is, by definition, the dense set of elements of $M$ which do not annihilate the previous function.

*Thus, upon identifying $\mathcal{C}_c(M /\!\!/ M_1, R)$ with $\mathcal{R} \otimes_{\mathbb{Z}} R$, the above dot-action is given on basis elements $mM_1 \in \Lambda_M$ $(m \in M)$ as follows*

$$w \bullet (mM_1) = \left(\frac{\delta_B(n_w m n_w^{-1})}{\delta_B(m)}\right)^{1/2} w(mM_1),$$

*and extended $R$-linearly to $\mathcal{R} \otimes_{\mathbb{Z}} R$.*

The last formulation shows that the dot-action is indeed well-defined and compatible with the algebraic structure. Define $c(m,n) := \delta_B(nmn^{-1})^{1/2}\delta_B(m)^{-1/2}$ for every $m \in M$ and any $n \in N$. Note that since the $\delta_B$ is trivial on the compact $M_1$ it factors through $\Lambda_M$, similarly the notation $c(m,n)$ factors then through the image of $(m,n)$ in $\Lambda_M \times W$. The following lemma will be used in the sequel.

**Lemma** III.10.0.4. *For any $m \in M$ and $n \in N$ with image $w$ in $W$, we have*

*(i) $c(m,n) = \prod_{\alpha \in \Phi^-_{\mathrm{red}} \cap w^{-1}(\Phi^+_{\mathrm{red}})} \delta_\alpha(m) \in q^{\mathbb{Z}}$.*

*(ii) If $m \in M^-$ then $c(m,n) \in q^{\mathbb{N}}$.*

*Proof.* (i) Recall that the modulus character $\delta_B \colon B = MU^+ \to q^{\mathbb{Z}}, mu \mapsto |\det \mathrm{Ad}_{\mathrm{Lie}(U^+)}(m)|$ (it is trivial on $U^+$)[22]. But as we have seen in Theorem II.2.5.1 one has $\mathrm{Lie}(\mathbf{U}^+) = \bigoplus_{\alpha \in \Phi^+_{\mathrm{red}}}(\mathrm{Lie}(\mathbf{U}_\alpha))$. Therefore,

$$\delta_B(mn) = \prod_{\alpha \in \Phi^+_{\mathrm{red}}} \delta_\alpha(m)$$

where $\delta_\alpha(m) := |\det \mathrm{Ad}_{\mathrm{Lie}(\mathbf{U}_\alpha)}(m)|$. Now, since $n\mathbf{U}_\alpha n^{-1} = \mathbf{U}_{w(\alpha)}$ one also have $n(\mathrm{Lie}(\mathbf{U}_\alpha))n^{-1} = \mathrm{Lie}(\mathbf{U}_{w(\alpha)})$, and so

$$\begin{aligned}
\delta_\alpha(m) &= |\det \mathrm{Ad}_{\mathrm{Lie}(\mathbf{U}_\alpha)}(m)| \\
&= |\det \mathrm{Ad}_{\mathrm{Lie}(\mathbf{U}_{w(\alpha)})}(nmn^{-1})| \\
&= \delta_{w(\alpha)}(nmn^{-1}) = \delta_{w^{-1}(\alpha)}(n^{-1}mn).
\end{aligned}$$

[22] Being an absolute value of a determinant on an $F$-vector space, the modulus function has clearly its values in $q^{\mathbb{Z}}$.

Accordingly,

$$\begin{aligned}
c(m,n) = \delta_B(nmn^{-1})^{1/2}\delta_B(m)^{-1/2} &= \frac{\prod_{\alpha\in\Phi^+_{\mathrm{red}}}\delta_\alpha(nmn^{-1})^{1/2}}{\prod_{\alpha\in\Phi^+_{\mathrm{red}}}\delta_\alpha(m)^{1/2}} \\
&= \frac{\prod_{\alpha\in\Phi^+_{\mathrm{red}}}\delta_{w^{-1}(\alpha)}(m)^{1/2}}{\prod_{\alpha\in\Phi^+_{\mathrm{red}}}\delta_\alpha(m)^{1/2}} \\
&= \frac{\prod_{\alpha\in\Phi^+_{\mathrm{red}}\cap w(\Phi^+_{\mathrm{red}})}\delta_{w^{-1}(\alpha)}(m)^{1/2}\cdot\prod_{\alpha\in\Phi^+_{\mathrm{red}}\cap w(\Phi^-_{\mathrm{red}})}\delta_{w^{-1}(\alpha)}(m)^{1/2}}{\prod_{\alpha\in\Phi^+_{\mathrm{red}}}\delta_\alpha(m)^{1/2}} \\
&= \frac{\prod_{\alpha\in\Phi^+_{\mathrm{red}}\cap w(\Phi^-_{\mathrm{red}})}\delta_{w^{-1}(\alpha)}(m)^{1/2}}{\prod_{\alpha\in\Phi^+_{\mathrm{red}}\setminus w(\Phi^+_{\mathrm{red}})}\delta_\alpha(m)^{1/2}} \\
&\overset{(1)}{=} \frac{\prod_{\alpha\in\Phi^+_{\mathrm{red}}\cap w(\Phi^-_{\mathrm{red}})}\delta_{w^{-1}(\alpha)}(m)^{1/2}}{\prod_{\alpha\in\Phi^+_{\mathrm{red}}\cap w(\Phi^-_{\mathrm{red}})}\delta_{w^{-1}(-\alpha)}(m)^{1/2}} \\
&= \prod_{\alpha\in\Phi^+_{\mathrm{red}}\cap w(\Phi^-_{\mathrm{red}})}\delta_{w^{-1}(\alpha)}(m) \\
&= \prod_{\alpha\in\Phi^-_{\mathrm{red}}\cap w^{-1}(\Phi^+_{\mathrm{red}})}\delta_\alpha(m)
\end{aligned}$$

we have used for (1): (i) the bijection

$$w^{-1}\colon \Phi^+_{\mathrm{red}}\cap w(\Phi^-_{\mathrm{red}}) \xrightarrow{\ \simeq\ } \Phi^-_{\mathrm{red}}\cap w^{-1}(\Phi^+_{\mathrm{red}}),$$

(ii) the equalities

$$\Phi^-_{\mathrm{red}}\cap w^{-1}(\Phi^+_{\mathrm{red}}) = -\left(\Phi^+_{\mathrm{red}}\cap w^{-1}(\Phi^-_{\mathrm{red}})\right) = -\left(\Phi^+_{\mathrm{red}}\setminus w^{-1}(\Phi^+_{\mathrm{red}})\right)$$

and (iii)

$$\begin{aligned}
\prod_{\alpha\in\Phi^+_{\mathrm{red}}\setminus w(\Phi^+_{\mathrm{red}})}\delta_\alpha(m)^{1/2} &= \prod_{\alpha\in-\left(\Phi^+_{\mathrm{red}}\setminus w^{-1}(\Phi^-_{\mathrm{red}})\right)}\delta_{w^{-1}(\alpha)}(m)^{1/2} \\
&\overset{(\delta_{-\alpha}=\delta_\alpha^{-1})}{=} \prod_{\alpha\in\Phi^+_{\mathrm{red}}\setminus w^{-1}(\Phi^-_{\mathrm{red}})}\delta_{w^{-1}(\alpha)}(m)^{-1/2}.
\end{aligned}$$

(ii) If $m\in M^-$, then one has $\omega(\det \mathrm{Ad}_{\mathrm{Lie}(\mathbf{U}_\alpha)}(m))\leq 0$ for any $\alpha\in\Phi^-_{\mathrm{red}}$. Hence, $c(m,n)\in q^{\mathbb{N}}$ for any $n\in N$. □

Let $(\mathcal{R}\otimes_{\mathbb{Z}}R)^{(W,\bullet)}$ denote the $R$-submodule of elements invariant under the dot-action. By (1) Lemma III.4.0.1 it is clear that $(\mathcal{R}\otimes_{\mathbb{Z}}R)^{(W,\bullet)}$ admits the following $R$-basis

$$\{r_m := \sum_{w\in W/W_m} w\bullet m = \sum_{w\in W/W_m} c(m,n_w)w(m)\colon m\in\Lambda^-_M\},$$

where, $n_w\in N\cap K$ is a representative for $w\in W$ and $W_m$ denotes the isotropy subgroup of $m$ in $W$ (for the initial action of the Weyl group, not the $\bullet$-action). Let $\mathcal{R}^{(W,\bullet)}$ denotes the $\mathbb{Z}$-submodule generated by the above family $\{r_m\colon m\in\Lambda^-_M\}$, by (ii) Lemma III.10.0.4

this is precisely the submodule of $\mathbb{Z}[\Lambda]$ of elements invariant under the $\bullet$-action.

**Lemma** III.10.0.5. *The image of $\dot{\mathcal{S}}_M^G$ lies in $(\mathcal{R} \otimes_{\mathbb{Z}} R)^{(W,\bullet)} = \mathcal{H}(M /\!\!/ M_1, R)^{(W,\bullet)}$, i.e.*

$$\dot{\mathcal{S}}_M^G \colon \mathcal{H}_K(\mathbb{Z}) \hookrightarrow \mathcal{R}^{(W,\bullet)} .$$

*Proof.* Let $w \in W$ represented by $n_w \in N \cap K$, Let $m \in M$. By the Lemmas III.10.0.2 and III.10.0.3, it suffices to show that $\delta_B(w \bullet m) = \delta_B(m)$ for all $m$ in the dense set of elements of $M$ which are regular[23] and semi-simple as elements in $G$ and all $w \in W$ (represented by $n_w \in N \cap K$). Therefore,

$$\begin{aligned}
\Delta(w \bullet m) &= c(m, n_w)\Delta(n_w m n_w^{-1}) && \text{(By definition)}\\
&= \delta_B(m)^{-1/2}\delta_B(n_w m n_w^{-1})^{1/2}\Delta(n_w m n_w^{-1})\\
&= \delta_B(m)^{-1/2}\delta_B(m)^{1/2}\Delta(m) && \text{[Car79, (23)]}\\
&= \Delta(m). \quad \square
\end{aligned}$$

In fact, the lemma above can be strengthened if we enlarge the coefficient ring by the inverse of $q$:

**Theorem** III.10.0.1. *The untwisted Satake transform induces a canonical isomorphism of $R$-algebras from $\mathcal{H}_K(R)$ to $(\mathcal{R} \otimes_{\mathbb{Z}} R)^{(W,\bullet)}$.*

*Proof.* Using the Lemmas III.10.0.1, III.10.0.2, III.10.0.3 and III.10.0.5, the remaining step for the proof of the above theorem, is to show that $h \mapsto \dot{\mathcal{S}}_M^G(h)$ is "upper triangular with respect to some total order, with invertible diagonals". This is proved in [HR10, §10.2] for the twisted Satake transform with coefficients in $\mathbb{C}$ which remains morally valid in our case as well although we express the total order in a slightly different way.

By Lemma III.10.0.1, to prove the theorem, it suffices to show that the composition of the following maps

$$\begin{array}{ccccc}
\mathcal{H}_K(R) & \longrightarrow & (\mathcal{R} \otimes_{\mathbb{Z}} R)^{(W,\bullet)} & \xrightarrow{\simeq} & (\mathcal{R} \otimes_{\mathbb{Z}} R)^{(W,\bullet)} \cdot v_{1,K}\\
h & \longmapsto & \dot{\mathcal{S}}_M^G(h) & \longmapsto & \dot{\mathcal{S}}_M^G(h) *_K v_{1,K}
\end{array}$$

is "upper triangular with respect to some total order, with invertible diagonals". The proof will be similar to the proof of Theorem III.8.0.1.

We recall that $\{h_x \colon x \in \Lambda_M^-\}$ forms a $\mathbb{Z}$-basis for $\mathcal{H}_K(\mathbb{Z})$ (§III.6), we also have a $R$-basis for $(\mathcal{R} \otimes_{\mathbb{Z}} R)^{(W,\bullet)} \cdot v_{1,K}$ given by $\{\sum_{w \in W/W_m} c(m, n_w) v_{w(m),K} \colon m \in \Lambda_M^-\}$. Fix an element

[23] Defined in footnote 21 on 96.

$x \in \Lambda_M^-$, hence

$$v_{1,K} *_K h_x = \dot{\mathcal{S}}_M^G(h_x) *_K v_{1,K} = \sum_{m \in \Lambda_M^-} c_{x,m} \left( \sum_{w \in W/W_m} c(m, n_w) v_{w(m),K} \right)$$

for some unique $c_{x,m} \in R$. Fix $m \in \Lambda_M^-$ such that $c_{x,m} \neq 0$, then $v_{1,K} *_I x(m) \neq 0$. Now, since $v_{1,K} *_I h_x$ can be non-zero only on the set $M_1 U^+ x K$, thus $m \in M_1 U^+ x K$[24], which implies that $U^+ m K \cap K x K \neq 0$.

The same proof of Lemma III.8.0.3 shows that $U^+ m K \cap K x K \neq 0$ implies $m \leq x$ in the Chevalley–Bruhat order: Indeed, using the decomposition $K = \sqcup_{w \in W} I^+ w I$ of Proposition III.7.0.2 and Lemma III.7.0.2 we see that $x \in U^+ m K = \cup_{w \in W} m U^+ w I$, thus $x \in U^+ m w_\circ I$ for some $w_\circ \in W$, so by Lemma III.8.0.3 one has $m w_\circ \leq x$. By definition of the Chevalley–Bruhat order if $x = s_1 \cdots s_{\ell(x)}$ is a reduced word for $x$ then $m w_\circ = s_{i_1} \cdots s_{i_{\ell(m w_\circ)}}$ is a subword of $x$. But by (4) Lemma III.4.0.1 one has $\ell(m w_\circ) = \ell(m) + \ell(w_\circ)$, hence $m = s_{i_1} \cdots s_{i_{\ell(m)}}$, i.e.

$$m \leq x.$$

We conclude that $c(x, m) = |U^+ m K \cap K x K| = 0$, unless $x \geq m$. Therefore, since the monoid $\Lambda_M^-$ is countable and every element $x \in \Lambda_M^-$ has only finitely many predecessors with respect to the Chevalley–Bruhat partial order, there exists a lexicographic (total) ordering $x_1, x_2, \cdots$ for the elements of $\Lambda_M^-$.

The above discussion shows that the matrix of the transformation $h \mapsto \dot{\mathcal{S}}_M^G(h)$ with respect to the bases $\{h_{x_i}\}_1^\infty$ and $\{x_i\}_1^\infty$ is upper triangular. Therefore, we obtain

$$\dot{\mathcal{S}}_M^G(h_x) *_I v_{1,K} = \sum_{m \in \Lambda_M^- :\, x \geq m} c_{x,m} \left( \sum_{w \in W/W_m} c(m, n_w) v_{w(m),K} \right).$$

We also remark that the diagonals $c_{x,x} c(x,1) = |U^+ x K \cap K x K|$, using the facts: $x \in \Lambda_M^-$, $K = (U^+ \cap K) M_1 (U^- \cap K)$, $x (U^+ \cap K) x^{-1} \subset (U^+ \cap K)$ and $x^{-1} (U^- \cap K) x \subset (U^- \cap K)$ together with $U^+ U^- \cap N = \{1\}$, we deduce that

$$c_{x,x} = |(U^+ \cap K) x M_1 (U^- \cap K)| = |K x K| \in q^{\mathbb{N}}.$$

This observation shows that the triangular matrix $(c_{x,m})_{x,m \in \Lambda_M^-}$ is indeed invertible once we extend the scalars from $\mathbb{Z}$ to $R$, and this shows that the homomorphism $h \mapsto \dot{\mathcal{S}}_M^G(h)$ induce a isomorphism of $R$-algebras

$$\mathcal{H}_K(R) \xrightarrow{\simeq} (\mathcal{R} \otimes_{\mathbb{Z}} R)^{(W, \bullet)}. \qquad \square$$

**Corollary** III.10.0.1. *The untwisted Satake homomorphism induces an isomorphism*

[24] We abuse notation, and use the letter $m$ for a class in $\Lambda_M^-$ and for a representative in $M^-$.

*of $\mathbb{Z}$-modules between*

$$\mathcal{H}_K(\mathbb{Z}) \xrightarrow{\sim} \mathcal{R}^{(W,\bullet)}.$$

*Proof.* By the Theorem III.10.0.1, It suffices to show that the image of $\mathcal{H}_K(\mathbb{Z})$ by the untwisted Satake homomorphism is precisely $\mathcal{R}^{(W,\bullet)}$, i.e. that each $r_m$, $m \in \Lambda_M^-$, is in the image of $\dot{\mathcal{S}}_M^G(\mathcal{H}_K(\mathbb{Z}))$. This claim is proven in [HV15, §7.14][25]. □

**Remark** III.10.0.4. *Consider the $\mathbb{C}$-linear map $\eta_B\colon \mathcal{R}\otimes_{\mathbb{Z}}\mathbb{C} \longrightarrow \mathcal{R}\otimes_{\mathbb{Z}}\mathbb{C}$, given on $\mathbb{C}$-basis elements $m \in \Lambda_M$ by*

$$\eta_B\colon m\otimes c \longmapsto \delta_B(m)^{1/2} m \otimes c.$$

*Observe that for any $m \in \Lambda_M$*

$$\eta_B\Big(\sum_{w\in W/W_m} w(m)\Big) = \sum_{w\in W/W_m} \delta(w(m))^{1/2} w(m) = \delta(m)^{1/2} \sum_{w\in W/W_m} w \bullet m.$$

*Hence, $\eta_B\colon \mathcal{R}\otimes_{\mathbb{Z}}\mathbb{C} \longrightarrow \mathcal{R}\otimes_{\mathbb{Z}}\mathbb{C}$ is actually an isomorphism of $\mathbb{C}$-algebras with $W$-action where the target is endowed with the $\bullet$-action (Definition III.10.0.1) and the source with the usual action. Let $\mathcal{S}_T^G\colon \mathcal{H}_K(\mathbb{C}) \to \mathcal{R}\otimes_{\mathbb{Z}}\mathbb{C}$ the "classical" untwisted Satake homomorphism [HR10, §9.2], then we have*

$$\eta_B \circ \mathcal{S}_T^G = \dot{\mathcal{S}}_T^G \otimes id_{\mathbb{C}}. \quad \square$$

## III.11 The center of the Iwahori–Hecke algebra

**Definition** III.11.0.1. *For each $m \in \Lambda_M^-$, define*

$$z_m := \dot{\Theta}_{\mathrm{Bern}}(r_m) \in \dot{\Theta}_{\mathrm{Bern}}((\mathcal{R}\otimes_{\mathbb{Z}} R)^{(W,\bullet)}) \subset \mathcal{H}_I(R),$$

*where, $W \bullet m$ denotes the orbit of $m$ under the twisted action of the Weyl group. We call the elements $z_m$ the untwisted Bernstein functions.*

We have

$$z_m = \sum_{m'\in W\bullet m} \dot{\Theta}_{m'} = \sum_{w\in W/W_m} c(m,n_w)\dot{\Theta}_{w(m)},$$

where, $W_m$ denotes the isotropy subgroup of $m$ in $W$.

**Proposition** III.11.0.1. *The Bernstein map $\dot{\Theta}_{\mathrm{Bern}}$ induces an embedding of algebras*

$$(\mathcal{R}\otimes_{\mathbb{Z}} R)^{(W,\bullet)} \hookrightarrow Z\left(\mathcal{H}_I(R)\right),$$

[25] We point the reader to the fact that definition we use for dominance is opposed to the one used by [HV15], their translation map is equal to $-\nu$ (see §3.2 *loc. cit.*).

*i.e. for each $m \in \Lambda_M^-$ the Iwahori–Hecke operator $z_m$ lies in the center of $\mathcal{H}_I(R)$.*

*Proof.* As we have seen in corollary III.9.0.1, the Iwahori–Hecke algebra $\mathcal{H}_I(R)$ has the following $R$-basis $\{\dot{\Theta}_m \otimes i_w \colon m \in \Lambda_M, w \in W\}$. Thus, to show that $z_m$ lies in the center for all $m \in \Lambda_M^-$, it suffices (Remark III.9.0.4) to show that it commutes with the elements $i_w$ for all $w \in W$. We formulate the different steps of the proof as lemmas:

**Lemma** III.11.0.1. *The $(\mathcal{R} \otimes_{\mathbb{Z}} R)$-submodule $(\mathcal{R} \otimes_{\mathbb{Z}} R) \cdot (v_1 *_I (\sum_W i_w)) \subset \mathcal{M}_I(R)$ is free of rank* 1.

*Proof.* This is a consequence of Lemma III.10.0.1, since $\sum_W i_w = \mathbf{1}_K$ by Proposition III.7.0.1. □

**Lemma** III.11.0.2. *For any $m \in \Lambda_M^-$, we have $\bar{z}_m := z_m *_I \mathbf{1}_K \in \mathcal{H}_K(R)$ and accordingly $[K : I]\mathbf{1}_K *_I \bar{z}_m = \bar{z}_m$*

*Proof.* We have

$$v_1 *_I z_m *_I \mathbf{1}_K = (\sum_{m' \in W \bullet m} m') \cdot v_1 *_I \mathbf{1}_K \in \mathcal{R}^{(W,\bullet)} \cdot v_{1,K} \in \mathcal{M}_I(R) *_I e_K$$

so by Theorem III.10.0.1, since $z_m \in \mathcal{R}^{(W,\bullet)}$, there exists a unique $\bar{z}_m \in \mathcal{H}_K(R)$ verifying

$$(\sum_{m' \in W \bullet m} m') \cdot v_1 *_I \mathbf{1}_K = (v_1 *_I \mathbf{1}_K) *_K \bar{z}_m = [K : I]^{-1}(v_1 *_I \mathbf{1}_K) *_I \bar{z}_m.$$

Now, since $\mathcal{M}_I(R)$ is a free $\mathcal{H}_I(R)$-module of rank 1, we must have

$$z_m *_I \mathbf{1}_K = [K : I]^{-1}\mathbf{1}_K *_I \bar{z}_m = \bar{z}_m \in \mathcal{H}_K(R).$$

In particular, $[K : I]\mathbf{1}_K *_I \bar{z}_m = \mathbf{1}_K *_K \bar{z}_m = \bar{z}_m$. □

**Lemma** III.11.0.3. *For any $m \in M$ and any reflection $s \in \mathcal{S}$:*

$$\dot{\Theta}_{m,s} := \dot{\Theta}_m *_I i_s - i_s *_I \dot{\Theta}_{w \bullet m} \in \dot{\Theta}_{\mathrm{Bern}}(\mathcal{R}) \otimes_{\mathbb{Z}} R.$$

*Proof.* Let $m \in M$. We want to apply [Ros15, Proposition 5.4.2]. Since in *loc. cit.* the notation $\Theta_m$ does not equal our $\dot{\Theta}_m$, we first compare the two to make things clear. Let $m_1, m_2 \in M^-$ verifying $m = m_1 - m_2$, hence

$$\begin{aligned}
\Theta_m &= \frac{[Im_2I : I]^{1/2}}{[Im_1I : I]^{1/2}} i_{m_1} *_I i_{m_2}^{-1} && \text{[Ros15, Definition 5.3.1]} \\
&= \frac{\delta_B(m_1)^{1/2}}{\delta_B(m_2)^{1/2}} i_{m_1} *_I i_{m_2}^{-1} && \text{Lemma III.7.0.4} \\
&= \delta_B(m)^{\frac{1}{2}} \dot{\Theta}_m.
\end{aligned}$$

We have for every reflexion $s \in \mathcal{S}$

$$\begin{aligned}\dot\Theta_{m,s} :&= \dot\Theta_m *_I i_s - i_s *_I \dot\Theta_{s\bullet m}\\ &= \dot\Theta_m *_I i_s - i_s *_I c(m,s)\dot\Theta_{s(m)}\\ &= \dot\Theta_m *_I i_s - i_s *_I \frac{\delta_B(s(m))^{\frac{1}{2}}}{\delta_B(m)^{\frac{1}{2}}}\dot\Theta_{s(m)}\\ &= \delta_B(m)^{-\frac{1}{2}}\left(\delta_B(m)^{\frac{1}{2}}\dot\Theta_m *_I i_s - i_s *_I \delta_B(s(m))^{\frac{1}{2}}\dot\Theta_{s(m)}\right)\\ &= q_m^{\frac{1}{2}}\left(\Theta_m *_I i_s - i_s *_I \Theta_{s(m)}\right).\end{aligned}$$

Therefore, applying [Ros15, Proposition 5.4.2] shows that

$$\dot\Theta_{m,s} \in \mathcal{H}_I(R) \cap \dot\Theta_{\mathrm{Bern}}(\mathcal{R}) \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm\frac{1}{2}}] \subset \dot\Theta_{\mathrm{Bern}}(\mathcal{R}) \otimes_{\mathbb{Z}} R. \quad \square$$

Let $s \in \mathcal{S}$ and $m \in \Lambda_M$, then by Lemma III.11.0.3

$$\dot\Theta_{z_m,s} := z_m *_I i_s - i_s *_I z_m = \sum_{w\in W/W_m} \dot\Theta_{w\bullet m} *_I i_s - i_s *_I \dot\Theta_{sw\bullet m} \in \dot\Theta_{\mathrm{Bern}}(\mathcal{R}) \otimes_{\mathbb{Z}} R$$

Let us compute the following,

$$\begin{aligned}\dot\Theta_{z_m,s} *_I \sum_{w\in W} i_w &= z_m *_I i_s *_I \mathbf{1}_K - i_s *_I z_m *_I \mathbf{1}_K && \text{(Proposition III.7.0.1)}\\ &= [IsI:I]\overline{z}_m - i_s *_I \overline{z}_m && (i_s * \mathbf{1}_K = [IsI:I]\mathbf{1}_K)\\ &= [IsI:I]\overline{z}_m - [IsI:I] i_s *_I (\mathbf{1}_K *_I \overline{z}_m) && \text{(Lemma III.11.0.2)}\\ &= [IsI:I]\overline{z}_m - [IsI:I]^2 \mathbf{1}_K *_I \overline{z}_m\\ &= 0 && \text{(Lemma III.11.0.2)}.\end{aligned}$$

Thus, by Lemma III.11.0.1, we see that $\dot\Theta_{z_m,s} = z_m *_I i_s - i_s *_I z_m = 0$. Finally, we deduce the desired commutativity for any $w = s_1 \cdots s_{\ell(w)}$ by induction on the length $\ell(w)$ in the Coxeter system $(W, \mathcal{S})$. $\square$

**Theorem** III.11.0.1. *The set $\{z_m : m \in \Lambda_M^-\}$ forms a basis for the $R$-module $Z(\mathcal{H}_I(R))$.*

*Proof.* We intend to use [Ros15, Proposition 6.3.1]. Rostami's proof which is heavily based on [Ros16], is valid only for $\mathbb{C}$-coefficients, and to our knowledge, it can be adapted to the best to give the same statement for $\mathbb{Q}$-coefficients. The untwisted definition of $\dot\Theta_{\mathrm{Bern}}$ allows to obtain the desired result with $R$-coefficients using the following trick:

Let $f \in Z(\mathcal{H}_I(R))$, by Rostami's [Ros15, Proposition 6.3.1], we can decompose it as

$$f = \sum_{m\in\Lambda_M^-} c_{f,m} z_m \quad (c_{f,m} \in \mathbb{C})$$

such that, $c_{f,m} \neq 0$ for only a finite set of $m \in J_f \subset \Lambda_M^-$. We know by (2) of Lemma

III.4.0.1, that for any $m \in J_f$ there exists a $m_\circ \in \Lambda_M^-$ such that

$$w(m) + m_\circ \in \Lambda_M^-$$

for all $w \in W$. Hence, for all $m \in J_f$

$$\begin{aligned} z_m &= \sum_{w \in W/W_m} c(m, n_w) \dot{\Theta}_{w(m)} \\ &= i_{m_\circ}^{-1} *_I \sum_{w \in W/W_m} c(m, n_w) i_{w(m)+m_\circ} && \text{Proposition III.9.0.1.} \end{aligned}$$

This shows that

$$i_{m_\circ} *_I f = \sum_{m \in J_f} c_{f,m} \sum_{w \in W/W_m} c(m, n_w) i_{w(m)+m_\circ}.$$

The function on the left hand side is a $R$-valued function, so

$$c_{f,m} c(m, n_w) = (i_{m_\circ} *_I f)\,(w(m) + m_\circ) \in R, \quad (\forall m \in J_f, \forall w \in W).$$

Therefore $c_{f,m} \in R$, since by Lemma III.10.0.4, we have $c(m, n_w) \in q^{\mathbb{N}}$. $\square$

**Corollary** III.11.0.1. *The homomorphism*

$$\dot{\Theta}_{Bern} \colon (\mathcal{R} \otimes_{\mathbb{Z}} R)^{(W,\bullet)} \longrightarrow Z\left(\mathcal{H}_I(R)\right),$$

*is an isomorphism of $R$-algebras.*

## III.12 Compatibility of the untwisted Satake and untwisted Bernstein isomorphisms

**Theorem** III.12.0.1. *The Satake and Bernstein untwisted isomorphisms are compatible, i.e., the following diagram is commutative:*

$$\begin{tikzcd}
\mathcal{H}_K(R) \arrow[dr, "\dot{\mathcal{S}}^G_M", "\simeq"'] & \\
& \mathcal{R}^{(W,\bullet)} \otimes_{\mathbb{Z}} R \simeq \mathcal{H}(M /\!\!/ M_1, R)^{(W,\bullet)} \arrow[dl, "\simeq"', "\dot{\Theta}_{\mathrm{Bern}}"] \\
Z\left(\mathcal{H}_I(R)\right) \arrow[uu, "-*_I \mathbf{1}_K", "\simeq"'] &
\end{tikzcd}$$

*Proof.* To our knowledge the first proof of the compatibility between the Satake and Bernstein isomorphism, i.e. commutativity of the diagram above is due to J.-F. Dat [Dat99, §3]. In *loc. cit.* a complete proof is provided only for split reductive groups. Here, we propose a different proof, following the one given for split groups in [HKP10, §4.6]: Let

$h \in \mathcal{H}_K(R)$, we have

$$\begin{aligned}
v_1 *_I h &= v_1 *_I e_K *_I h && (e_K *_I h = h)\\
&= [K:I](v_1 *_I e_K) *_K h && (\text{passing from } *_I \text{ to } *_K)\\
&= [K:I]\dot{\mathcal{S}}_M^G(h)\cdot(v_1 *_I e_K) && (\text{Definition of } \dot{\mathcal{S}}_M^G)\\
&= [K:I](\dot{\mathcal{S}}_M^G(h)\cdot v_1) *_I e_K &&\\
&= [K:I](v_1 *_I \dot{\Theta}_{Bern}(\dot{\mathcal{S}}_M^G(h))) *_I e_K && (\text{Definition of } \dot{\Theta}_{\mathrm{Bern}})\\
&= v_1 *_I \dot{\Theta}_{Bern} \circ \dot{\mathcal{S}}_M^G(h) *_I \mathbf{1}_K &&
\end{aligned}$$

Hence, $h = \dot{\Theta}_{Bern} \circ \dot{\mathcal{S}}_M^G(h) *_I \mathbf{1}_K = \mathbf{1}_K *_I \dot{\Theta}_{Bern} \circ \dot{\mathcal{S}}_M^G(h)$. □

**Remark** III.12.0.1. *The convolution products in $\mathcal{H}_K(R)$ and $\mathcal{H}_I(R)$ are not the same, and the normalized vertical map in Theorem III.12.0.1 is indeed a morphism of algebras. Write the vertical map $Z(\mathcal{H}_I(R)) \to \mathcal{H}_K(R) \colon h \mapsto \bar{h}$. Let $h_1, h_2 \in Z(\mathcal{H}_I(R))$, we have $\overline{h_1 *_I h_2} = \bar{h}_1 *_K \bar{h}_2$, since:*

$$\begin{aligned}
(h_1 * \mathbf{1}_K) *_K (h_2 * \mathbf{1}_K) &= \frac{1}{[K:I]} h_1 *_I e_K *_K h_2 *_I e_K\\
&= \frac{1}{[K:I]} h_1 *_I h_2 *_I \mathbf{1}_K *_I \mathbf{1}_K\\
&= (h_1 *_I h_2) *_I \mathbf{1}_K.
\end{aligned}$$ □

Consequently, combing the above theorem with Corollary III.10.0.1 we get

**Corollary** III.12.0.1. *The Satake and Bernstein untwisted $\mathbb{Z}$-isomorphisms are compatible, i.e., the following diagram (of $\mathbb{Z}$-modules) is commutative:*

$$\begin{tikzcd}
\mathcal{H}_K(\mathbb{Z}) \arrow[dr, "\dot{\mathcal{S}}_M^G", "\simeq"'] & \\
& \mathcal{R}^{(W,\bullet)} \arrow[dl, "\dot{\Theta}_{\mathrm{Bern}}", "\simeq"'] \\
\dot{\Theta}_{\mathrm{Bern}}(\mathcal{R}^{(W,\bullet)}) \arrow[uu, "- *_I \mathbf{1}_K", "\simeq"'] &
\end{tikzcd}$$

**Remark** III.12.0.2. *The $\mathbb{Z}$-algebra $\dot{\Theta}_{\mathrm{Bern}}(\mathcal{R}^{(W,\bullet)}) \subset Z(\mathcal{H}_I(R))$ may not be in $\mathcal{H}_I(\mathbb{Z})$.*

## III.13 The ring of $\mathbb{U}$-operators

We introduce here two special commutative subrings of $\mathcal{H}_I(R)$, but before this, let us look at the homomorphism $\dot{\Theta}_{\mathrm{Bern}}$ from another viewpoint.

The group $\Lambda_M$ is clearly the Grothendieck group of the monoid $\Lambda_M^-$[26], therefore

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Ring}}(\mathcal{R}, \mathcal{H}_I(R)) &\simeq \mathrm{Hom}_{\mathrm{Group}}(\Lambda_M, \mathcal{H}_I(R)^\times) \\ &\simeq \mathrm{Hom}_{\mathrm{Monoid}}(\Lambda_M^-, \mathcal{H}_I(R)^\times).\end{aligned}$$

We have seen that for each $m \in \Lambda_M^-$, the Iwahori–Hecke operator $i_m$ is invertible in $\mathcal{H}_I(R)$. By proposition III.9.0.1, we see that the untwisted Bernstein map $\dot{\Theta}_{\mathrm{Bern}} \in \mathrm{Hom}_{\mathrm{Ring}}(\mathcal{R}, \mathcal{H}_I(R))$ is precisely the homomorphism of rings that corresponds to the homomorphism of monoids

$$(m \mapsto i_m) \in \mathrm{Hom}_{\mathrm{Monoid}}(\Lambda_M^-, \mathcal{H}_I(R)^\times)$$

via the above natural isomorphisms.

**Definition** III.13.0.1. *Define $\mathbb{U}^+ := \dot{\Theta}_{Bern}(\mathcal{R})$, this is equivalently the ring generated by the elements $i_m$ and $i_m^{-1}$ for all $m \in \Lambda_M^-$ (see Proposition III.9.0.1). Define also the ring of $\mathbb{U}$-operators to be*

$$\mathbb{U} := \dot{\Theta}_{Bern}(\mathbb{Z}[\Lambda_M^-]) \subset \mathcal{H}_I(\mathbb{Z}).$$

**Theorem** III.13.0.1. *The subring $\dot{\Theta}_{\mathrm{Bern}}(\mathcal{R}^{(W,\bullet)})\,[\mathbb{U}^+]$ of $\mathcal{H}_I(R)$, spanned by $\mathbb{U}^+$ and $\dot{\Theta}_{\mathrm{Bern}}(\mathcal{R}^{(W,\bullet)})$, is integral over $\dot{\Theta}_{\mathrm{Bern}}(\mathcal{R}^{(W,\bullet)}) \subset Z\left(\mathcal{H}_I(R)\right)$.*

*Proof.* Let $m \in \mathcal{R}$, it suffices to consider the polynomial

$$P_m(X) := \prod_{w \in W/W_m} (X - \dot{\Theta}_{\mathrm{Bern}}(w \bullet m)) \in \dot{\Theta}_{\mathrm{Bern}}(\mathcal{R}^{(W,\bullet)})[X],$$

where, $W_m$ denotes the isotropy subgroup of $m$ in $W$. This ends the proof, since by Corollary III.11.0.1 we have $\dot{\Theta}_{\mathrm{Bern}}(\mathcal{R}^{(W,\bullet)}) \subset Z\left(\mathcal{H}_I(R)\right)$. □

**Remark** III.13.0.1. *Note that by definition $P_m$ is the minimal polynomial annihilating $\dot{\Theta}_m$.*

## III.14 A $\mathbb{U}$-structure on $\mathbb{Z}[G/K]$

Recall that Proposition III.5.0.1 gives another interpretation of the Iwahori–Hecke algebra $\mathcal{H}_I(\mathbb{Z})$, namely being naturally isomorphic to the (opposite) ring of intertwiners $\mathrm{End}_G\mathbb{Z}[G/I]$ of the right regular representation $\mathbb{Z}[G/I]$ of $G$ associated to $I$:

$$\mathcal{H}_I(\mathbb{Z}) \xrightarrow{\simeq} \mathrm{End}_G(\mathbb{Z}[G/I])])^{\mathrm{opp}},$$

[26]It is universal for homomorphisms of the monoid $\Lambda_M^-$ into groups.

here, the superscript *opp* indicates the opposite ring. Likewise, there is a natural isomorphism of rings

$$\mathcal{H}_K(\mathbb{Z}) \xrightarrow{\simeq} \mathrm{End}_{\mathbb{Z}[G]}(\mathbb{Z}[G/K])])^{\mathrm{opp}},$$

here, we may drop the superscript "opp", since these rings are commutative.

We would like the ring $\mathbb{U}$ to act on $\mathcal{C}_c(G/K,\mathbb{Z})$, although it lives in $\mathcal{H}_I(\mathbb{Z})$. More precisely, we will construct an homomorphism of rings

$$\mathbb{U} \longrightarrow \mathrm{End}_B(\mathbb{Z}[G/K])])^{\mathrm{opp}}.$$

## III.14.1 Section

The inclusion $I \subset K$ induces a surjective map

$$\mathcal{C}_c(G/I,\mathbb{Z}) \twoheadrightarrow \mathcal{C}_c(G/K,\mathbb{Z}), \qquad f \mapsto f *_I \mathbf{1}_K.$$

The same inclusion induces also natural embeddings

$$\mathcal{C}_c(G/K,\mathbb{Z}) \hookrightarrow \mathcal{C}_c(G/I,\mathbb{Z}), \qquad \mathcal{H}_K(\mathbb{Z}) \hookrightarrow \mathcal{H}_I(\mathbb{Z}).$$

The composition of the above maps

$$\mathcal{C}_c(G/K,\mathbb{Z}) \hookrightarrow \mathcal{C}_c(G/I,\mathbb{Z}) \twoheadrightarrow \mathcal{C}_c(G/K,\mathbb{Z}),$$

is multiplication by $[K:I]$. We fix a section of the natural projection $G/I \to G/K$, as follows:

$$s\colon G/K \xrightarrow{\simeq} B/(B\cap K) \hookrightarrow G/I,$$

the bijectivity of the first map follows from the Iwasawa decomposition $G = BK$ (Proposition III.7.0.3), while the injectivity of the second map follows from the computation of the kernel of the natural projection $B \to G/I$ in the following lemma:

**Lemma** III.14.1.1. *We have $B\cap I = B\cap K$.*

*Proof.* On the one hand, applying [HV15, (ii) §6.8] [27] we get:

$$B\cap K = (M\cap K)(U^+\cap K) \text{ and } B\cap I = (M\cap I)(U^+\cap I).$$

On the other hand, by corollary II.3.9.2 we have $U^+\cap K = I^+ = U^+\cap I$, and by Lemma II.3.9.2 we have $M\cap K = M_1 = M\cap I$. This shows the lemma. □

[27] Take the set $\Omega$ in *loc. cit.* to be the fixed special vertex $a_\circ$ first, and then to be the alcove $\mathfrak{a}$.

The above section induces a map, also denoted by $s$

$$s\colon \mathcal{C}_c(G/K,\mathbb{Z}) \longrightarrow \mathcal{C}_c(G/I,\mathbb{Z}),$$

defined on the basis functions by $\mathbf{1}_{bK} \mapsto \mathbf{1}_{bI}$, for all $b \in B$. The map $s$, is actually a retraction of $- *_I \mathbf{1}_K\colon \mathcal{C}_c(G/I,\mathbb{Z}) \twoheadrightarrow \mathcal{C}_c(G/K,\mathbb{Z})$, indeed for all $b \in B$ we have $s(\mathbf{1}_{bK}) *_I \mathbf{1}_K = \mathbf{1}_{bI} *_I \mathbf{1}_K = \mathbf{1}_{bK}$, that is the composition

$$\mathcal{C}_c(G/K,\mathbb{Z}) \xrightarrow{s} \mathcal{C}_c(G/I,\mathbb{Z}) \twoheadrightarrow \mathcal{C}_c(G/K,\mathbb{Z})$$

is the identity.

## III.14.2 Pairing

We define an "excursion pairing"

$$\mathcal{C}_c(G/K,\mathbb{Z}) \quad \times \quad \mathcal{H}_I(\mathbb{Z}) \longrightarrow \mathcal{C}_c(G/K,\mathbb{Z}),$$

as follows:

$$x \bullet f := (s(x) *_I f) *_I \mathbf{1}_K.$$

This is clearly bilinear in both variables. For $x = \mathbf{1}_{bK}$ and $f = \mathbf{1}_{IgI}$, we get (using Lemma III.5.0.1)

$$x \bullet f = \mathbf{1}_{bIgI} *_I \mathbf{1}_K = \sum_{i \in I/I\cap gIg^{-1}} \mathbf{1}_{bigI} *_I \mathbf{1}_K = \sum_{i \in I/I\cap gIg^{-1}} \mathbf{1}_{bigK},$$

where, we have used $IgI = \sqcup_{i \in I/I\cap gIg^{-1}} igI$.

**Lemma** III.14.2.1. *($\mathbb{U}$ action on $\mathbb{Z}[G/K]$) The "excursion pairing" when restricted to $\mathbb{U}$ defines a right action*

$$\mathcal{C}_c(G/K,\mathbb{Z}) \quad \times \quad \mathbb{U} \longrightarrow \mathcal{C}_c(G/K,\mathbb{Z}).$$

*Proof.* We need to verify, that for all $r_1 = m_1M_1, r_2 = m_2M_1 \in \Lambda_M^-$, we have

$$(\mathbf{1}_{gK} \bullet i_{r_1}) \bullet i_{r_2} = \mathbf{1}_{gK} \bullet (i_{r_1} *_I i_{r_2}) = \mathbf{1}_{gK} \bullet i_{r_1+r_2}.$$

Recall that $\Theta_r = i_r$ for any $r \in \Lambda_M^-$, and the braid relation

$$i_{r_1} *_I i_{r_2} = i_{r_1+r_2}, \quad r_1, r_2 \in \Lambda_M^- \text{ in } \mathcal{H}_I(\mathbb{Z}). \tag{III.4}$$

We have for $? \in \{1,2\}$ (using the Iwahori factorization of $I$)

$$\begin{aligned} Im_?I &= I^+M_1I^-m_?I \\ &= I^+m_?M_1(m_?^{-1}I^-m_?)I && \text{(Lemma III.7.0.3)} \\ &= I^+m_?I \\ &= \sqcup_{b'\in I^+/m_?I^+m_?^{-1}} b'm_?I \end{aligned}$$

now the left hand side of (III.4) is given by (see Example III.5.0.1)

$$\begin{aligned} i_{r_1} *_I i_{r_2} &= \sum_{b'\in[I^+/m_1I^+m_1^{-1}]} \mathbf{1}_{b'm_1I} *_I \sum_{b''\in[I^+/m_2I^+m_2^{-1}]} \mathbf{1}_{b''m_2I} \\ &= \sum_{\substack{b'\in[I^+/m_1I^+m_1^{-1}] \\ b''\in[I^+/m_2I^+m_2^{-1}]}} \mathbf{1}_{b'm_1b''m_2I} \end{aligned}$$

reusing (III.4) we get

$$\sum_{\substack{b'\in[I^+/I^+\cap m_1I^+m_1^{-1}] \\ b''\in[I^+/m_2I^+m_2^{-1}]}} \mathbf{1}_{b'm_1b''m_2I} = \sum_{b'''\in[I^+/(m_1m_2)I^+(m_1m_2)^{-1}]} \mathbf{1}_{b'''m_1m_2I} \tag{III.5}$$

Let $b \in B$ be any representative of $gK$, thus

$$(\mathbf{1}_{gK} \bullet i_{r_1}) \bullet i_{r_2} = ((\mathbf{1}_{bI} *_I i_{r_1}) *_I \mathbf{1}_K) \bullet i_{r_2}.$$

On the other hand we also have

$$\begin{aligned} \mathbf{1}_{bI} *_I i_{r_1} &= \mathbf{1}_{bI} *_I \mathbf{1}_{Im_1I} \\ &= \mathbf{1}_{bIm_1I} && \text{(Lemma III.5.0.1)} \\ &= \sum_{b'\in[I^+/m_1I^+m_1^{-1}]} \mathbf{1}_{bb'm_1I} \end{aligned}$$

Note that we have specified representatives $b'$ in $I^+$ to ensure that the resulting $bb'm_1$ stay in $B$. Therefore,

$$\begin{aligned}
(\mathbf{1}_{gK} \bullet i_{r_1}) \bullet i_{r_2} &= \left( \sum_{b' \in [I^+/m_1 I^+ m_1^{-1}]} \mathbf{1}_{bb'm_1 I} *_I \mathbf{1}_K \right) \bullet i_{r_2} \\
&= \left( \sum_{b' \in [I^+/m_1 I^+ m_1^{-1}]} \mathbf{1}_{bb'm_1 K} \right) \bullet i_{r_2} \\
&= \sum_{b' \in [I^+/m_1 I^+ m_1^{-1}]} \mathbf{1}_{bb'm_1 K} \bullet i_{r_2} \\
&= \sum_{\substack{b' \in [I^+/m_1 I^+ m_1^{-1}] \\ b'' \in [I^+/m_2 I^+ m_2^{-1}]}} \mathbf{1}_{bb'm_1 b'' m_2 I} *_I \mathbf{1}_K \\
&= \sum_{b''' \in [I^+/(m_1 m_2) I^+ (m_1 m_2)^{-1}]} \mathbf{1}_{bb''' m_1 m_2 I} *_I \mathbf{1}_K \qquad \text{(use (III.5))} \\
&= \mathbf{1}_{gK} \bullet i_{r_1 + r_2}.
\end{aligned}$$

□

**Lemma** III.14.2.2. *The action of $\mathbb{U}$ on $\mathcal{C}_c(G/K, \mathbb{Z})$ is faithful.*

*Proof.* For any $m_1, \cdots m_r \in \Lambda_M^-$ distinct, and $s_1, \cdots, s_r \in \mathbb{Z}$, we have

$$\begin{aligned}
\mathbf{1}_K *_I (\sum_i s_i \mathcal{U}_{m_i} \bullet 1_K) &= \mathbf{1}_K *_I \left( \sum_i s_i \mathbf{1}_I *_I i_{m_i} \right) *_I \mathbf{1}_K \\
&= \mathbf{1}_K *_I \left( \sum_i s_i i_{m_i} \right) *_I \mathbf{1}_K \\
&= \sum_i s_i \frac{|K \cap m_i K m_i^{-1}|_I}{|I \cap m_i I m_i^{-1}|_I} \mathbf{1}_{K m_i K} \qquad \text{(Lemma III.5.0.1)}.
\end{aligned}$$

But, by the Cartan decomposition (Proposition III.4.0.1) we know that the $(h_{m_i})_i$ are linearly independent since all $m_i$ belongs to $\Lambda_M^-$. Therefore, if we assume that

$$\sum s_i \mathcal{U}_{m_i} \bullet \mathbf{1}_K = \mathbf{1}_K,$$

then we must have

$$s_i = \begin{cases} 1 & \text{If } m_i = 1 \in \Lambda_M^-, \\ 0 & \text{If } m_i \neq 1 \in \Lambda_M^-. \end{cases}$$

□

**Corollary** III.14.2.1. *The action of $\mathbb{U}$ on $\mathcal{C}_c(G/K, \mathbb{Z})$ induces an embedding of rings*

$$\mathbb{U} \xhookrightarrow{\varphi} \mathrm{End}_B \mathcal{C}_c(G/K, \mathbb{Z})^{\mathrm{opp}} \subset \mathrm{End}_M \mathcal{C}_c(G/K, \mathbb{Z})^{\mathrm{opp}}.$$

*Proof.* The $B$-equivariance is a straightforward consequence of the formula

$$\mathbf{1}_{bK} \bullet i_m = \sum_{b' \in [I^+/m I^+ m^{-1}]} \mathbf{1}_{bb'm_1 K},$$

for $b \in B$ and $m \in M^-$. The injectivity was proven in Lemma III.14.2.2. □

**Lemma** III.14.2.3. *For every $m \in \Lambda_M^-$, let $\mathbf{P}_m = \mathbf{U}_m^+ \rtimes \mathbf{L}_m$ be the largest parabolic subgroup of $\mathbf{G}$ relative to which $m$ is anti-dominant. In this case, $\mathbf{P}_m$ is a* semi-standard *parabolic subgroup containing the minimal parabolic subgroup $\mathbf{B}$. Set $\mathbb{U}_m$ for the subring of $\mathbb{U}$ corresponding to the monoid $\mathbb{N} \cdot m$, i.e. $\langle i_{n \cdot m} \colon n \in \mathbb{N} \rangle$. Then $\varphi$ induces a embedding of rings*

$$\mathbb{U}_m \hookrightarrow \mathrm{End}_{P_m} \mathcal{C}_c(G/K, \mathbb{Z})^{\mathrm{opp}} \subset \mathrm{End}_B \mathcal{C}_c(G/K, \mathbb{Z})^{\mathrm{opp}}.$$

*Proof.* The $P_m$-equivariance is a consequence of (i) the Iwasawa decomposition $G = PK$ (Proposition III.7.0.3) and (ii) the formula

$$\begin{aligned} \mathbf{1}_{pK} \bullet i_m &= \sum_{p' \in [I^+/mI^+m^{-1}]} \mathbf{1}_{pp'mK} \\ &\overset{\text{Lemma V.2.3.2}}{=} \sum_{p' \in [U_m^+ \cap I^+/U_m^+ \cap mI^+m^{-1}]} \mathbf{1}_{pp'mK}. \end{aligned}$$

□

**Remark** III.14.2.1. *Let $h \in Z(\mathcal{H}_I(\mathbb{Z}))$, $\bar{h} = h *_I \mathbf{1}_K$, and $u \in \mathbb{U}$. If $h$ and $u$ are $\neq \mathbf{1}_I$, we have*

$$(\mathbf{1}_{gK} *_K \bar{h}) \bullet u \neq (\mathbf{1}_{gK} \bullet u) *_K \bar{h}.$$

*In particular, we see that the action of $\mathbb{U}$ viewed in $\mathrm{End}_B\, \mathcal{C}_c(G/K, \mathbb{Z})$ does not commute with the action of $\mathrm{End}_G\, \mathcal{C}_c(G/K, \mathbb{Z})$.*

**Remark** III.14.2.2. *Here, the only properties of $K$ that we have used are: $I \subset K$ and $B \cap I = B \cap K$. This holds for many parahoric subgroups, and for larger groups.*

## III.14.3 Integrality

**Theorem** III.14.3.1. *For every $u \in \mathbb{U} \subset \mathrm{End}_B \mathcal{C}_c(G/K, \mathbb{Z})^{\mathrm{opp}}$, there exists a monic polynomial $Q_u(X) = \sum \bar{h}_k X^k \in \mathrm{End}_G \mathcal{C}_c(G/K, \mathbb{Z})[X]$*

$$\sum \bar{h}_k u^k = 0, \;\; \textit{in } \mathrm{End}_B \mathcal{C}_c(G/K, \mathbb{Z})[X].$$

*Proof.* Let $u \in \mathbb{U} \subset \mathcal{H}_I(\mathbb{Z})$ and $\mathbf{1}_{bK} \in \mathcal{C}_c(G/K, R)$ with $b \in B$. By Theorem III.13.0.1, there exist a monic polynomial $P_u(X) = \sum_k X^k h_k \in \dot{\Theta}_{Bern}(\mathcal{R}^{(W,\bullet)})[X]$ (set $h_n = \mathbf{1}_I$, the unit in $\mathcal{H}_I(R)$) having $u$ as a root. Set $\bar{h}_k := h_k *_I \mathbf{1}_K \in \mathcal{H}_K(\mathbb{Z})$ (Corollary III.12.0.1). We have

$$\begin{aligned}
\sum_k (\mathbf{1}_{bK} \bullet u^k) *_K \bar{h}_k &= \sum_k (\mathbf{1}_{bK} \bullet u^k) *_K (h_k *_I \mathbf{1}_K) \\
&= |K|^{-1} \sum_k (\mathbf{1}_{bK} \bullet u^k) *_I (h_k *_I \mathbf{1}_K) \\
&= [K:I]^{-1} \sum_k ((\mathbf{1}_{bI} *_I u^k) *_I \mathbf{1}_K) *_I (h_k *_I \mathbf{1}_K) \\
&= [K:I]^{-1} \mathbf{1}_{bI} *_I \left( \sum_k u^k *_I h_k \right) *_I \mathbf{1}_K *_I \mathbf{1}_K \\
&= \mathbf{1}_{bI} *_I P_u(u) *_I \mathbf{1}_K \\
&= 0 \in \mathcal{C}_c(G/K, \mathbb{Z}).
\end{aligned}$$

This holds for all $b \in B$, therefore:

$$\sum \bar{h}_k \circ u^k = 0 \in \operatorname{End}_B \mathcal{C}_c(G/K, \mathbb{Z}),$$

with $\bar{h}_k \in \operatorname{End}_G \mathcal{C}_c(G/K, \mathbb{Z})$. $\square$

### III.14.4 Example $\mathbf{GL}_2$

Let us make some verification by hand for $\mathbf{G} = \mathbf{GL}_{2,\mathbb{Z}_p}$. Let $T$ be the diagonal matrices, $B$ the Borel subgroup of upper triangular matrices, $K$ the maximal open compact subgroup $\mathbf{G}(\mathbb{Z}_p)$ and $I$ the corresponding Iwahori subgroup $\left\{ \begin{pmatrix} * & * \\ p* & * \end{pmatrix} \subset K \right\}$. Set $g_? = \begin{pmatrix} p & ? \\ & 1 \end{pmatrix}$ for any $? \in \mathbb{Z}_p$. We have

$$Ig_0I = \bigsqcup_{a \in \mathbb{Z}_p/p\mathbb{Z}_p} \begin{pmatrix} p & a \\ & 1 \end{pmatrix} I = \bigsqcup_{a \in \mathbb{Z}_p/p\mathbb{Z}_p} g_a I.$$

Thus, for any $b \in B$

$$\begin{aligned}
\mathbf{1}_{bK} \bullet \mathbf{1}_{Ig_0I} &= (\mathbf{1}_{bI} *_I \mathbf{1}_{Ig_0I}) *_I \mathbf{1}_K \\
&= \sum_{a \in \mathbb{Z}_p/p\mathbb{Z}_p} \mathbf{1}_{bg_aI} *_I \mathbf{1}_K \\
&= \sum_{a \in \mathbb{Z}_p/p\mathbb{Z}_p} \mathbf{1}_{bg_aK}
\end{aligned}$$

therefore,

$$\mathbf{1}_{bK} \bullet \mathbf{1}_{Ig_0I} = \mathbf{1}_{bK} *_K T_p - \mathbf{1}_{bg_0'K},$$

where, $T_p = \mathbf{1}_{Kg_0K}$ and $g'_? := \begin{pmatrix} 1 & ? \\ & p \end{pmatrix}$. Hence,

$$\begin{aligned}
\mathbf{1}_{bK} \bullet \mathbf{1}_{Ig_0I}^2 &= (\mathbf{1}_{bK} \bullet \mathbf{1}_{Ig_0I}) \bullet \mathbf{1}_{Ig_0I} \\
&= \left( \sum_{a \in \mathbb{Z}_p/p\mathbb{Z}_p} \mathbf{1}_{bg_aK} \right) \bullet \mathbf{1}_{Ig_0I} \\
&= \sum_{a \in \mathbb{Z}_p/p\mathbb{Z}_p} (\mathbf{1}_{bg_aK} \bullet \mathbf{1}_{Ig_0I}) \\
&= \sum_{a \in \mathbb{Z}_p/p\mathbb{Z}_p} \mathbf{1}_{bg_aK} *_K T_p - \mathbf{1}_{bg_ag'_0K} \\
&= (\mathbf{1}_{bK} \bullet \mathbf{1}_{Ig_0I}) *_K T_p - p\mathbf{1}_{pbK} \\
&= (\mathbf{1}_{bK} \bullet \mathbf{1}_{Ig_0I}) *_K T_p - p\mathbf{1}_{bK} *_K S_p
\end{aligned}$$

where, $S_p = \mathbf{1}_{pK} \in \mathcal{C}_c(G/K, \mathbb{Z})$. where, we have used the fact that $g_a g'_0 K = pK$ for any $a \in \mathbb{Z}_p$. In conclusion, we see that the $\bullet$-action defines an operator $u \in \operatorname{End}_{\mathbb{Z}[B]} \mathcal{C}_c(G/K, \mathbb{Z})$ corresponding to $\mathbf{1}_{Ig_0I} \in \mathcal{H}_I(\mathbb{Z})$ such that

$$u^2 - T_p u + pS_p = 0,$$

where this time, $T_p \in \operatorname{End}_{\mathbb{Z}[G]} \mathcal{C}_c(G/K, \mathbb{Z})$ (resp. $S_p \in \operatorname{End}_{\mathbb{Z}[G]} \mathcal{C}_c(G/K, \mathbb{Z})$) is the Hecke operator corresponding to $Kg_0K$ (resp. $Kg_0g'_0K$)

## III.15 $\widetilde{\mathbb{U}}$-operators for $\widetilde{K}$ aka Geometric $\mathbb{U}$-operators

In this section we will show how the whole above story adapts when we replace the parahoric subgroups $K$ and $I$ by the $G^1$-fixators[28] $\widetilde{K}$ and $\widetilde{I}$ of $a_\circ$ and $\mathfrak{a}$ respectively. We will then define the corresponding ring of $\mathbb{U}$-operators, this ring is interesting because it is "visible" in the sense that it appears as a ring of geometric operators on the extended building of $\mathbf{G}$ (see §V.1.3).

### III.15.1 Digression on $G^1$-stabilizers

To every facet in the extended building $\mathcal{F} \times V_G$, we may associate an open compact subgroup closely related to the parahoric subgroup $K_\mathcal{F}$ (definition II.3.9.1), this is $\widetilde{K}_\mathcal{F}$ the $G$-fixator of the facet $\mathcal{F} \times V_G \subset \mathcal{B}(\mathbf{G}, F)_{\text{ext}}$. By definition of the action of $G$ on the affine

[28] Recall that we use the word "fixator" to indicate pointwise stabilizers.

space $V_G$ we see that $\widetilde{K}_{\mathcal{F}} = P_{\mathcal{F}} \cap G^1$, i.e. it is also the $G^1$-fixators of the facet $\mathcal{F}$.

Assume now that the facet verifies $a_\circ \in \overline{\mathcal{F}} \subset \bar{\mathfrak{a}}$, which is equivalent to $I \subset K_{\mathcal{F}} \subset K$. Consider the composition map

$$\widetilde{K}_{\mathcal{F}} \to \widetilde{K} \twoheadrightarrow \widetilde{K}/K.$$

By [Ros15, Lemma 4.2.1], on the one hand this map is surjective since $\widetilde{K}_{\mathcal{F}}$ contains $\widetilde{I}$, on the other hand it induces a group isomorphism

$$\widetilde{K}_{\mathcal{F}}/K_{\mathcal{F}} \simeq \widetilde{K}/K,$$

since $\widetilde{K}_{\mathcal{F}} \cap K = K_{\mathcal{F}}$. Since also $[\widetilde{K} : K][K : K_{\mathcal{F}}] = [\widetilde{K} : \widetilde{K}_{\mathcal{F}}][\widetilde{K}_{\mathcal{F}} : K_{\mathcal{F}}]$, so

$$[\widetilde{K} : \widetilde{K}_{\mathcal{F}}] = [K : K_{\mathcal{F}}].$$

By [HR10, Proposition 11.1.4], we have an inclusion $M_1 \subset \widetilde{K}$ and it induces a bijection $M^1/M_1 \simeq \widetilde{K}/K$. Using [HR10, Proposition 8.0.1] we see that $M^1 = \widetilde{I} \cap M$ hence $M^1 = \widetilde{K}_{\mathcal{F}} \cap M$, $M^1 \subset \widetilde{K}_{\mathcal{F}}$ and

$$\widetilde{K}_{\mathcal{F}} = M^1 K_{\mathcal{F}}.$$

**Lemma** III.15.1.1. *The open compact subgroups $\widetilde{K}_{\mathcal{F}}$ admits the following factorization*

$$\widetilde{K}_{\mathcal{F}} = U^+_{\mathcal{F}} U^-_{\mathcal{F}} U^+_{\mathcal{F}} M^1.$$

*Proof.* This is a consequence of three ingredients: the equality $\widetilde{K}_{\mathcal{F}} = M^1 K_{\mathcal{F}}$, the Iwahori factorization for $K_{\mathcal{F}}$ as in Proposition II.3.9.2 and finally the fact that $M^1$ normalizes $U^+_{\mathcal{F}}$ and $U^-_{\mathcal{F}}$. □

**Lemma** III.15.1.2. *The inclusion $M \hookrightarrow G$ induces an equality $M_1 = M \cap G_1$ and accordingly an inclusion*

$$(\Lambda_M)_{tor} \hookrightarrow (\Lambda_G)_{tor} \ ,$$

*compatible with $\kappa_M$ and $\kappa_G$.*

*Proof.* By definition we have $M \cap G_1 \subset M_1$. The other inclusion follows from

$$M_1 \overset{\text{Lemma II.3.9.2}}{=} M \cap K \subset M \cap G_1.$$

Consider the composition of the two following maps

$$M^1 \hookrightarrow G^1 \twoheadrightarrow G^1/G_1 \ .$$

But $M \cap G_1 = M_1$, which shows that the induced map $(\Lambda_M)_{\text{tor}} \hookrightarrow (\Lambda_G)_{\text{tor}}$ is an inclusion.

**Remark** III.15.1.1. *As we have seen in §II.3.9.2, the torsion parts are precisely $(\Lambda_M)_{tor} = M^1/M_1$ and $(\Lambda_G)_{tor} = G^1/G_1$, hence applying the above lemma shows that $M_1 = M^1 \cap G_1$.*

**Remark** III.15.1.2. *By the whole above discussion we see that,*

$$\begin{tikzcd} & (\Lambda_G)_{tor} = 0 \arrow[rr, Leftrightarrow] & & G^1 = G_1 \arrow[d, Rightarrow] \\ & (\Lambda_M)_{tor} = 0 \arrow[r, Leftrightarrow] & M^1 = M_1 \arrow[r, Leftrightarrow] & \widetilde{K}_{\mathcal{F}} = K_{\mathcal{F}}. \end{tikzcd}$$

*where $\mathcal{F} \subset \mathcal{B}(\mathbf{G}, F)_{\mathrm{red}}$ is any facet containing in its closure $\mathfrak{a}_\circ$. For example, if $\mathbf{G}$ is unramified or semi-simple simply connected then $\Lambda_M$ is torsion free so $\widetilde{K} = K$ (see [HR10, Corollaries 11.1.2. & 11.1.7.]).*

## III.15.2 Averaging homomorphisms

**Remark** III.15.2.1. *From now on, we will use the nomenclature 'geometric', to differentiate between the context of parahorics and $G^1$-stabilizers, e.g. $\mathcal{H}_{\widetilde{K}}(\mathbb{Z})$ will be called the geometric spherical Hecke algebra, $\mathcal{H}_{\widetilde{I}}(\mathbb{Z})$ the geometric Iwahori–Hecke algebra*

We have a bijection

$$\widetilde{I}\backslash G/\widetilde{I} \simeq \widetilde{W}_{\mathrm{aff}}.$$

where $\widetilde{W}_{\mathrm{aff}}$ is the extended affine Weyl group $N/M^1 \simeq \underline{\Lambda}_M \rtimes W$ (§II.3.8), which is also bijective to the double coset $\widetilde{W}_{\mathrm{aff}} \xrightarrow{\sim} M^1U^+\backslash G/\widetilde{I}$ (proof similar to Lemma III.8.0.1 and Remark III.8.0.2)

**Remark** III.15.2.2. *Since $\mathfrak{a}_\circ$ is special, the canonical injection*

$$N \cap \widetilde{K}/M \cap \widetilde{K} \hookrightarrow W$$

*is an isomorphism. Therefore, we may and will assume when needed that every representative in $N$ of an element $w \in W = N/M$ lies in $\widetilde{K}$, such a representative is determined up to multiplication by $M^1 = M \cap \widetilde{K}$.*

Likewise, we have a bijection

$$\widetilde{K}\backslash G/\widetilde{K} \simeq \underline{\Lambda}_M^- = M^-/M^1.$$

The above two bijections yield $R$-basis for the geometric Iwahori–Hecke (resp. geometric Hecke algebras):

$$\{\widetilde{i}_w := \mathbf{1}_{\widetilde{I}w\widetilde{I}}, w \in \widetilde{W}_{\mathrm{aff}}\}, \ (\text{ resp. } \{\widetilde{h}_m := \mathbf{1}_{\widetilde{K}m\widetilde{K}}, m \in \underline{\Lambda}_M^-\}).$$

Set

$$q_{\text{tors}} := |(\Lambda_M)_{\text{tor}}| = |\widetilde{I}/I| = |\widetilde{K}/K|, R_{\text{t}} := R \otimes_{\mathbb{Z}} \mathbb{Z}[q_{\text{tors}}^{\pm 1}].$$

**Lemma** III.15.2.1. *The "averaging" map $\iota\colon \mathcal{H}_I(R_t) \to \mathcal{H}_{\widetilde{I}}(R_t)$ defined on basis elements*

$$\mathbf{1}_{IgI} \mapsto \frac{1}{[\widetilde{I}:I]} \sum_{h\in(\Lambda_M)_{tor}} \mathbf{1}_{IghI}$$

*is a surjective homomorphism of algebras.*

*Proof.* An $R_{\text{t}}$ basis of $\mathcal{H}_{\widetilde{I}}(R_{\text{t}})$ is given by elements $\mathbf{1}_{\widetilde{I}g\widetilde{I}}$ for all $g = mw \in \underline{\Lambda}_M \rtimes W$. We have $\widetilde{I} = \sqcup_{h\in(\Lambda_M)_{\text{tor}}} hI$, hence[29]

$$\begin{aligned}
\widetilde{I}mw\widetilde{I} &= \cup_{h',h\in(\Lambda_M)_{\text{tor}}} Ihmwh'I \\
&= \cup_{h',h\in(\Lambda_M)_{\text{tor}}} Imww^{-1}h''wh'I && m^{-1}hm = h'' \in M^1 \\
&= \cup_{h',h''\in(\Lambda_M)_{\text{tor}}} Imwh''h'I && \text{Lemma III.15.2.2} \\
&= \cup_{h\in(\Lambda_M)_{\text{tor}}} ImwhI.
\end{aligned}$$

Using the decomposition $G = \sqcup_{(m',w')\in\Lambda_M\ltimes W} Im'w'I$, one shows that for any fixed pair $mw \in \underline{\Lambda}_M \rtimes W$ the map $\Lambda_M)_{\text{tor}} \ni h \mapsto ImwhI$ is injective. Hence,

$$\iota\left(\mathbf{1}_{ImwI}\right) = \frac{1}{|(\Lambda_M)_{\text{tor}}|} \sum_{h\in(\Lambda_M)_{\text{tor}}} \mathbf{1}_{ImwhI} = \mathbf{1}_{\widetilde{I}mw\widetilde{I}}.$$

For the rest we refer to [Ros15, Proposition 4.2.3, Lemma 5.1.1 & Remark 5.1.2]. □

**Lemma** III.15.2.2. *The Weyl group $W$ acts trivially on $(\Lambda_M)_{tor} = M^1/M_1$.*

*Proof.* Let $w \in W$ with representative $n_w \in N \cap K$, so for any $m \in M^1$ we have

$$w(mM_1) = n_w m n_w^{-1} M_1,$$

but since $M^1$ is normal in $N$ we have $n_w m n_w^{-1} \in M^1$. We have already seen in §III.15.1 that the inclusion $M^1 \hookrightarrow \widetilde{K}$ induces the following surjective map

$$M^1 \twoheadrightarrow \widetilde{K}/K$$

with kernel $M^1 \cap K = M_1$. Hence $n_w m n_w^{-1} K = n_w m K$, but $m$ normalizes $K$ (since $K$ is normal in $\widetilde{K}$), so

$$n_w m n_w^{-1} K = m m^{-1} n_w m K = mK,$$

this shows that $mM_1 = n_w m n_w^{-1} M_1$ and proves the lemma. □

[29] We abuse notation and use the same letters for classes and representatives, which is not harmful for our computations.

## III.15.3 Geometric untwisted homomorphisms

All of what we have proved previously in the case $(I, K)$, can be proven mutatis mutandis, with slight modifications for the pair $(\widetilde{I}, \widetilde{K})$.

### III.15.3.1 Geometric untwisted Bernstein homomorphism

For any ring $R$, define as in §III.8 the $\mathcal{H}_{\widetilde{I}}(R)$-module $\mathcal{M}_{\widetilde{I}}(R) = \mathcal{C}_c(M^1U^+\backslash G/\widetilde{I}, R)$. There is a natural identification[30] between $R[\underline{\Lambda}_M]$ and the relative Hecke algebra $\mathcal{C}_c(M /\!\!/ M^1, R)$ that allows us to endow the $\mathcal{H}_{\widetilde{I}}(R)$-module $\mathcal{M}_{\widetilde{I}}(R)$ with a left $R[\underline{\Lambda}_M]$-action as follows: define for every $\psi \in \mathcal{M}_{\widetilde{I}}(R)$ and $r \in R[\underline{\Lambda}_M]$:

$$r \cdot \psi(g) := \int_M r(a)\psi(a^{-1}g)d\mu_{M^1}(a) \quad (\forall g \in G)$$

here, $d\mu_{M^1}(a)$ is the Haar measure on $M$ giving $M^1$ volume 1.

There is a canonical bijection between the extended affine Weyl group $\widetilde{W}_{\mathrm{aff}} = N/M^1 \simeq \underline{\Lambda}_M \rtimes W$. Therefore, the set $\{\widetilde{v}_x := \mathbf{1}_{M^1U^+x\widetilde{I}} \colon x \in \widetilde{W}_{\mathrm{aff}}\}$ gives an $R$-basis for the module $\mathcal{M}_{\widetilde{I}}(R)$. It is straightforward to adapt the proof of Proposition III.8.0.1 and get the rules: For every $w \in W$ and $m \in \underline{\Lambda}_M$, we have

1. $\widetilde{v}_1 *_{\widetilde{I}} \widetilde{i}_w = \widetilde{v}_w$,
2. $m \cdot \widetilde{v}_w = \widetilde{v}_{mw}$,
3. $\widetilde{v}_m *_{\widetilde{I}} \widetilde{i}_w = \widetilde{v}_{mw}$,
4. $\widetilde{v}_1 *_{\widetilde{I}} \widetilde{i}_m = \widetilde{v}_m$ if $m \in (\underline{\Lambda}_M)^-$.

Therefore, on the one hand the $R[\underline{\Lambda}_M]$-module is a free module of rank $|W|$, with canonical basis $\{\widetilde{v}_w, w \in W\}$, on the other hand it is free of rank 1 as a $\mathcal{H}_{\widetilde{I}}(R)$ (proof similar to corollary III.8.0.2) with canonical generator $\widetilde{v}_1$. This yields an embedding of $R$-algebras (the geometric untwisted Bernstein homomoprhism)

$$\begin{aligned} \dot{\widetilde{\Theta}}_{\mathrm{Bern}} \colon R &\hookrightarrow \mathcal{H}_{\widetilde{I}}(R) \\ m &\mapsto \dot{\widetilde{\Theta}}_m, \end{aligned}$$

characterized by the property: $m \cdot \widetilde{v}_1 = \widetilde{v}_1 *_{\widetilde{I}} \dot{\widetilde{\Theta}}_m$, for every $m \in R[\underline{\Lambda}_M]$. This induces a decomposition

$$R[\underline{\Lambda}_M] \otimes \mathcal{H}^0_{\widetilde{I}}(R) \xrightarrow{\simeq} \mathcal{H}^0_{\widetilde{I}}(R),$$

[30] This is a consequence of the fact that $M^1$ is the unique maximal compact subgroup of M.

where $\mathcal{H}^0_{\widetilde{I}}(R) := \mathcal{C}_c(\widetilde{K} /\!\!/ \widetilde{I}, R)$, we thus obtain an $R$-basis for $\mathcal{H}_{\widetilde{I}}(R)$

$$\{\dot{\widetilde{\Theta}}_m *_{\widetilde{I}} \widetilde{i}_w \colon m \in \underline{\Lambda}_M, w \in W\}.$$

Using the above discussion, we can express explicitly the geometric untwisted Bernstein homomorphism, indeed for any $m \in \underline{\Lambda}_M$, then

$$\dot{\widetilde{\Theta}}_m = \widetilde{i}_{m_1} *_{\widetilde{I}} (\widetilde{i}_{m_2})^{-1},$$

for any $m_1, m_2 \in \underline{\Lambda}_M^-$ satisfying $m = m_1 - m_2$. Using the averaging map of Lemma III.15.2.1, we see that for any $m \in \underline{\Lambda}_M$ we have

$$\iota(\dot{\Theta}_m) = \iota(i_{m_1}) *_{\widetilde{I}} \iota(i_{m_2})^{-1} = \dot{\widetilde{\Theta}}_{\iota(m)},$$

here $\iota(m)$ (by abuse of notation) also denotes the image of $m$ by the canonical surjection $\Lambda_M \twoheadrightarrow \underline{\Lambda}_M$, and it preserves dominance by definition (Compare with [Ros15, Lemma 5.3.1]).

### III.15.3.2 Geometric untwisted Bernstein homomorphism

Identically to §III.10, one can consider the bi-$(R[\underline{\Lambda}_M], \mathcal{H}_{\widetilde{K}}(R))$-module

$$\mathcal{M}_{\widetilde{K}}(R) := \mathcal{C}_c(M^1U^+\backslash G/\widetilde{K}, R).$$

It is free of rank 1 as a $R[\underline{\Lambda}_M]$-module with canonical generator $\widetilde{v}_{1,K} = \mathbf{1}_{M^1U^+\widetilde{K}}$, this yields a geometric untwisted Satake homomorphism

$$\dot{\widetilde{\mathcal{S}}}^G_M \colon \mathcal{H}_{\widetilde{K}}(R) \hookrightarrow R[\underline{\Lambda}_M].$$

**Remark** III.15.3.1. *We can naturally identify $\mathcal{H}_{\widetilde{K}}(\mathbb{Q})$ with the two-sided ideal $e_{\widetilde{K}} *_K \mathcal{H}_K(\mathbb{Q}) *_K e_{\widetilde{K}} \subset \mathcal{H}_K(\mathbb{Q})$, where $e_{\widetilde{K}}$ is the idempotent $|\widetilde{K} : K|^{-1}\mathbf{1}_{\widetilde{K}}$ (see Lemma III.5.0.3). In a similar way, $\mathcal{M}_{\widetilde{K}}(\mathbb{Q})$ can also be naturally identified with $\mathcal{M}_K(\mathbb{Q}) *_I e_{\widetilde{K}} \subset \mathcal{M}_K(\mathbb{Q})$, and the right action of $\mathcal{H}_{\widetilde{K}}(\mathbb{Q})$ will correspond to the right action of $e_{\widetilde{K}} *_K \mathcal{H}_K(\mathbb{Q}) *_K e_{\widetilde{K}}$.*

Computations similar to Lemma III.10.0.2 shows that

$$\dot{\widetilde{\mathcal{S}}}^G_M = \left(\dot{\mathcal{S}}^G_M\right)_{|\mathcal{H}_{\widetilde{K}}(R)}.$$

**Theorem** III.15.3.1. *The Satake transform induces a canonical isomorphism of $R$-algebras from $\mathcal{H}_{\widetilde{K}}(R)$ to $R[\underline{\Lambda}_M]^{(W,\bullet)}$.*

*Proof.* This theorem is of course no surprise [Sat63], or [Car79]. But now that we are here, let us derive it from what we have proven for the special maximal parahoric subgroup $K$. By Remark III.15.3.1 above, identify $\mathcal{H}_{\widetilde{K}}(\mathbb{Q})$ with $e_{\widetilde{K}} *_K \mathcal{H}_K(\mathbb{Q}) *_K e_{\widetilde{K}} \subset \mathcal{H}_K(\mathbb{Q})$. A direct application of Theorem III.10.0.1 shows then that the transform $\dot{\mathcal{S}}^G_M$ induces a canonical isomorphism of $R$-algebras between $\mathcal{H}_{\widetilde{K}}(R)$ and its image in $\mathcal{C}_c(M \mathbin{/\!\!/} M_1, R)^{(W,\bullet)}$, this latter image is precisely $\mathcal{C}_c(M \mathbin{/\!\!/} M^1, R)^{(W,\bullet)}$ by an argument similar to the proof of Theorem III.10.0.1. Accordingly, we get a commutative diagram

$$\begin{tikzcd}
\mathcal{H}_{\widetilde{K}}(R) \arrow[r, "\dot{\widetilde{\mathcal{S}}}^G_M", "\simeq"'] \arrow[d, hook] & \mathcal{C}_c(M \mathbin{/\!\!/} M^1, R)^{(W,\bullet)} \arrow[r, "\simeq"] \arrow[d, hook] & R[\underline{\Lambda}_M]^{(W,\bullet)} \arrow[d, hook] \\
\mathcal{H}_K(R) \arrow[r, "\dot{\mathcal{S}}^G_M", "\simeq"'] & \mathcal{C}_c(M \mathbin{/\!\!/} M_1, R)^{(W,\bullet)} \arrow[r, "\simeq"] & R[\Lambda_M]^{(W,\bullet)}
\end{tikzcd} \qquad \square$$

We forget the $R$-algebra structure, we still get a commutative diagram of $\mathbb{Z}$-modules

$$\begin{tikzcd}
\mathcal{H}_{\widetilde{K}}(R) \arrow[r, "\dot{\widetilde{\mathcal{S}}}^G_M", "\simeq"'] \arrow[d, hook] & R[\underline{\Lambda}_M]^{(W,\bullet)} \arrow[d, hook] \\
\mathcal{H}_K(R) \arrow[r, "\dot{\mathcal{S}}^G_M", "\simeq"'] & R[\Lambda_M]^{(W,\bullet)}
\end{tikzcd}$$

**Theorem** III.15.3.2. *The Satake transform induces a canonical isomorphism of $\mathbb{Z}$-modules*

$$\mathcal{H}_{\widetilde{K}}(\mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}[\underline{\Lambda}]^{(W,\bullet)}$$

*where, $\mathbb{Z}[\underline{\Lambda}]^{(W,\bullet)}$ denotes the $\mathbb{Z}$-module of elements of $\mathbb{Z}[\underline{\Lambda}]$ that are $W$-invariant.*

*Proof.* Initially, using the previous isomorphism theorem one shows the claim for the $\mathbb{Z}$-module structure, then deduce that it must respect the $\mathbb{Z}$-algebra structure. $\square$

**Remark** III.15.3.2. *Using the 'averaging' map, we can invert the vertical maps in the above diagram and still get a commutative diagram*

$$\begin{tikzcd}
\mathcal{H}_{\widetilde{K}}(R) \arrow[r, "\dot{\mathcal{S}}^G_M", "\simeq"'] & \mathcal{C}_c(M \mathbin{/\!\!/} M^1, R)^{(W,\bullet)} \arrow[r, "\simeq"] & R[\underline{\Lambda}_M]^{(W,\bullet)} \\
\mathcal{H}_K(R) \arrow[r, "\dot{\mathcal{S}}^G_M", "\simeq"'] \arrow[u, two heads] & \mathcal{C}_c(M \mathbin{/\!\!/} M_1, R)^{(W,\bullet)} \arrow[r, "\simeq"] \arrow[u, two heads] & R[\Lambda_M]^{(W,\bullet)} \arrow[u, two heads]
\end{tikzcd}$$

*here, the right vertical arrow is the map induced by the natural quotient map $\Lambda_M \twoheadrightarrow \underline{\Lambda}_M$ and the middle vertical arrow is the corresponding one, it is given by*

$$\mathbf{1}_{mM_1} \longmapsto \frac{1}{|(\Lambda_M)_{tor}|} \sum_{h \in (\Lambda_M)_{tor}} \mathbf{1}_{mhM_1}.$$

*Recall that since $M_1$ and $M^1$ are kernels, they are normal in $M$, so we have $M^1 m M^1 = mM^1$ and $M_1 m M_1 = mM_1$.*

### III.15.4 Compatibility

**Definition** III.15.4.1. *For each $m \in \underline{\Lambda}_M^-$, define*

$$\widetilde{z}_m := \dot{\widetilde{\Theta}}_{\mathrm{Bern}}\left(\sum_{m' \in W \bullet m} m'\right) \in \dot{\widetilde{\Theta}}_{\mathrm{Bern}}(\mathbb{Z}[\underline{\Lambda}_M]^{(W,\bullet)}) \subset \mathcal{H}_{\widetilde{I}}(R),$$

*where, $W \bullet m$ denotes the orbit of $m$ under the twisted action of the Weyl group. We call the elements $\widetilde{z}_m$ the geometric untwisted Bernstein functions.*

**Proposition** III.15.4.1. *The homomorphism*

$$\dot{\widetilde{\Theta}}_{Bern} \colon R[\underline{\Lambda}_M]^{(W,\bullet)} \longrightarrow Z\left(\mathcal{H}_{\widetilde{I}}(R)\right),$$

*is an isomorphism of $R$-algebras.*

*Proof.* Combine [Ros15, Remark 5.1.2 & Lemma 6.1.1] and [Lus89, Proposition 3.11]. □

We then get to a compatibility result similar to Theorem III.12.0.1:

**Theorem** III.15.4.1. *The Satake and Bernstein untwisted geometric isomorphisms are compatible in the sense that the following diagram is commutative:*

$$\begin{tikzcd}
\mathcal{H}_{\widetilde{K}}(R) \arrow[drr, "\dot{\widetilde{\mathcal{S}}}{}^G_M", "\simeq"'] & & \\
& & R[\underline{\Lambda}]^{(W,\bullet)} \simeq \mathcal{H}(M /\!\!/ M^1, R)^{(W,\bullet)} \arrow[dll, "\simeq"', "\dot{\widetilde{\Theta}}_{\mathrm{Bern}}"] \\
Z\left(\mathcal{H}_{\widetilde{I}}(R)\right) \arrow[uu, "-*_{\widetilde{I}}\mathbf{1}_{\widetilde{K}}", "\simeq"'] & &
\end{tikzcd}$$

**Corollary** III.15.4.1. *The Satake and Bernstein untwisted $\mathbb{Z}$-isomorphisms are compatible, i.e., the following diagram (of $\mathbb{Z}$-modules) is commutative:*

$$\begin{tikzcd}
\mathcal{H}_{\widetilde{K}}(\mathbb{Z}) \arrow[drr, "\dot{\widetilde{\mathcal{S}}}{}^G_M", "\simeq"'] & & \\
& & \mathbb{Z}[\underline{\Lambda}_M]^{(W,\bullet)} \arrow[dll, "\simeq"', "\dot{\widetilde{\Theta}}_{\mathrm{Bern}}"] \\
\dot{\Theta}_{\mathrm{Bern}}(\mathbb{Z}[\underline{\Lambda}_M]^{(W,\bullet)}) \arrow[uu, "-*_{\widetilde{I}}\mathbf{1}_{\widetilde{K}}", "\simeq"'] & &
\end{tikzcd}$$

### III.15.5 Geometric $\widetilde{\mathbb{U}}$-operators

Using the fact that $\widetilde{I} \subset \widetilde{K}$ and $B \cap \widetilde{K} = B \cap \widetilde{I} = M^1 I^+$, we get section of the natural projection $G/\widetilde{I} \to G/\widetilde{K}$, as follows:

$$\widetilde{s}\colon G/\widetilde{K} \xrightarrow{\simeq} B/(B \cap \widetilde{K}) \hookrightarrow G/\widetilde{I}.$$

this induces a map,

$$\widetilde{s}\colon \mathcal{C}_c(G/\widetilde{K}, \mathbb{Z}) \longrightarrow \mathcal{C}_c(G/\widetilde{I}, \mathbb{Z}),$$

defined on the basis functions by $\mathbf{1}_{b\widetilde{K}} \mapsto \mathbf{1}_{b\widetilde{I}}$, for all $b \in B$. The map $\widetilde{s}$ is a retraction of $- *_{\widetilde{I}} \mathbf{1}_{\widetilde{K}}\colon \mathcal{C}_c(G/\widetilde{I}, R) \twoheadrightarrow \mathcal{C}_c(G/\widetilde{K}, R)$.

**Remark** III.15.5.1. *On the level of the extended building, the injection $G/\widetilde{K} \hookrightarrow G/\widetilde{I}$ induces a $B$-equivariant embedding of the $G$-orbit of $(a_\circ, 0)$ into the $G$-orbit of $(\mathfrak{a}, 0)$.*

Define $\widetilde{\mathbb{U}}$ and $\widetilde{\mathbb{U}}^+$ to be the rings $\dot{\widetilde{\Theta}}_{\text{Bern}}(\mathbb{Z}[\underline{\Lambda}_M^-])$ and $\dot{\widetilde{\Theta}}_{\text{Bern}}(\mathbb{Z}[\underline{\Lambda}_M])$ respectively. In a similar way to what we have done in §III.14.2, define an "excursion pairing"

$$\mathcal{C}_c(G/\widetilde{K}, R) \quad \times \quad \mathcal{H}_{\widetilde{I}}(R) \longrightarrow \mathcal{C}_c(G/\widetilde{K}, R),$$

as follows:

$$x \bullet f := (\widetilde{s}(x) *_I f) *_{\widetilde{I}} \mathbf{1}_{\widetilde{K}}.$$

When restricted to $\widetilde{\mathbb{U}}^+$, this pairing defines a ring right action

$$\mathcal{C}_c(G/\widetilde{K}, \mathbb{Z}) \quad \times \quad \widetilde{\mathbb{U}} \longrightarrow \mathcal{C}_c(G/\widetilde{K}, \mathbb{Z}).$$

Same arguments for the proofs of Corollary III.14.2.1 and Theorem III.13.0.1 yield

**Theorem** III.15.5.1. *The subring $\dot{\Theta}_{\text{Bern}}(\mathbb{Z}[\underline{\Lambda}_M]^{(W,\bullet)})\left[\widetilde{\mathbb{U}}^+\right]$ of $\mathcal{H}_{\widetilde{I}}(R)$ is integral over $\dot{\Theta}_{\text{Bern}}(\mathbb{Z}[\underline{\Lambda}_M]^{(W,\bullet)})$, and the action of $\widetilde{\mathbb{U}}$ on $\mathcal{C}_c(G/\widetilde{K}, \mathbb{Z})$ induces an embedding of rings*

$$\widetilde{\mathbb{U}} \longrightarrow \mathrm{End}_B \mathcal{C}_c(G/\widetilde{K}, \mathbb{Z})^{\mathrm{opp}}.$$

**Corollary** III.15.5.1. *For every $u \in \widetilde{\mathbb{U}} \subset \mathrm{End}_B \mathcal{C}_c(G/\widetilde{K}, \mathbb{Z})^{\mathrm{opp}}$, there exists a monic polynomial $Q_u(X) = \sum \widetilde{h}_k X^k \in \mathrm{End}_G \mathcal{C}_c(G/\widetilde{K}, \mathbb{Z})[X]$*

$$\sum \widetilde{h}_k u^k = 0, \ \text{in } \mathrm{End}_B \mathcal{C}_c(G/\widetilde{K}, \mathbb{Z}).$$

# CHAPTER IV

# SEED RELATIONS AND HECKE POLYNOMIALS

Assume for this section that $\mathbf{G}$ is unramified (quasi-split over $F$ and split over an unramified extension of $F$), which is equivalent to the existence of a reductive $\mathcal{O}_F$-model $\mathcal{G}$ of $\mathbf{G}$. As noted in Remark III.15.1.2, in this situation one has $\widetilde{K} = K$, i.e. $K$ is the special maximal open compact subgroup fixing the vertex $a_\circ$. Assume further that $K$ is the hyperspecial maximal open compact subgroup $\mathcal{G}(\mathcal{O}_F)$.

By [GD70b, XXVI 7.15] and Lang's theorem [Lan56], $\mathcal{G}$ is quasi-split over $\mathcal{O}_F$: there is a Borel pair $\mathcal{T} \subset \mathcal{B} \subset \mathcal{G}$. We may and will assume that $\mathbf{S}$ extends to the maximal split $\mathcal{O}_F$-subtorus $\mathcal{S}$ of $\mathcal{T}$. Set $\mathbf{T} \subset \mathbf{B}$ for the base change $\mathcal{T}_F \subset \mathcal{B}_F$, respectively. Let $\mathbf{U}^+$ be the unipotent radical of $\mathbf{B}$.

The Bruhat-Tits translation homomorphism $\nu_N \colon N \colon \to (X_*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{R}) \rtimes W$ we have fixed in Proposition II.3.2.1 is normalized such that

$$\nu_N(\varpi^\lambda) = -\lambda \quad \text{(see Lemma II.3.2.2)}.$$

With the notation of the previous chapters, we have $Z_{\mathbf{G}}(\mathbf{S}) = \mathbf{T} = \mathcal{T}_F$ (this centralizer was denoted by $\mathbf{M}$ before) and as in Remark III.15.1.2 [1]:

$$T_1 = T^1 = \mathcal{T}(\mathcal{O}_F) = \ker \nu_N = \ker \kappa_T.$$

It is possible to embed $X_*(\mathbf{S})$ into $T$ in two natural ways. Our convention will be to identify $\lambda \in X_*(\mathbf{S})$ with $\varpi^\lambda$. Using this identification we have

$$\Lambda_T := T/T_1 \simeq X_*(\mathbf{T})_F \simeq X_*(\mathbf{S}).$$

Let $\Phi^+$ be the set of $\mathbf{B}$-positive roots, the one that appears in Lie($\mathbf{B}$). We say that $\lambda \in X_*(\mathbf{S})$ is $\mathbf{B}$-dominant if $\langle \lambda, \alpha \rangle \geq 0$ for all $\alpha \in \Phi^+$. Let $\overline{\mathcal{C}} \subset \mathcal{A}_{\text{ext}}$ denotes the closed vectorial chamber corresponding to the Borel $B$. Thus, an element $t = \varpi^\lambda := \lambda(\varpi)$ for $\lambda \in X_*(\mathbf{S})$ is antidominant (as defined in §III.4), if and only if $\lambda \in X_*(\mathbf{S}) \cap \overline{\mathcal{C}}$, if and only if $\lambda$ is $\mathbf{B}$-dominant, since

$$\langle \nu_N(t), \alpha \rangle = -\langle \lambda, \alpha \rangle \leq 0, \qquad \forall \alpha \in \Phi^+.$$

For any extension $E$ of $F$, let $\mathcal{M}(E)$ be the set of $\mathbf{G}(E)$-conjugacy classes of (algebraic group) cocharacters $\mathbb{G}_{m,E} \to \mathbf{G}_E$. The Cartan decomposition (Proposition III.4.0.1) yields the following identification

$$\mathcal{M}(F) \simeq K\backslash G/K,$$

given by $[\lambda] \mapsto K\varpi^\lambda K$, for $\lambda \in X_*(\mathbf{S})$ is a representative of $[\lambda] \in \mathcal{M}(F)$.

## IV.1 Langlands dual group

Let $\Gamma_{un} = \mathrm{Gal}(F^{un}/F) \simeq \mathrm{Gal}(\overline{k}_F/k_F)$. As before, we let $\sigma \in \Gamma_{un}$ be the arithmetic Frobenius of $F$. The group $\mathbf{G}$ split over $F^{un}$ [GD70b, XXVI 7.15]. We consider a Langlands dual group of $\mathbf{G}$ with respect to $\Gamma_{un}$. This group sits in the following short exact sequence

$$1 \longrightarrow \widehat{\mathbf{G}} \longrightarrow {}^L\mathbf{G} \longrightarrow \Gamma \longrightarrow 1,$$

and every choice of épinglage $(\widehat{\mathbf{B}}, \widehat{\mathbf{T}}, (e_\alpha))$[2] yields a splitting of the above exact sequence. We fix a $\Gamma_{un}$-invariant épinglage [Kot84b, §1] thus ${}^L\mathbf{G} = \widehat{\mathbf{G}} \rtimes \Gamma_{un}$.

[1]Note that in this unramified case, $\mathbf{T}$ splits over the completion of $F^{un}$ denoted previously by $L$. Thus, the Kottwitz homomorphism defined in §II.3.9.2 takes the simpler form

$$\kappa_{\mathbf{T}} \colon \mathbf{T}(L) \to X_*(\mathbf{T}).$$

[2]Here, for each simple root $\alpha$ of $\widehat{\mathbf{T}}$, $e_\alpha$ is a nonzero element of the root vector space $\mathrm{Lie}(\widehat{\mathbf{G}})_\alpha$.

The $\Gamma_{un}$-equivariant isomorphism $X_*(\mathbf{T}) \simeq X^*(\widehat{\mathbf{T}})$ induces a canonical identification between the $\Gamma_{un}$-groups $W(\mathbf{G}_{\overline{F}}, \mathbf{T})$ and the Weyl group $W(\widehat{\mathbf{G}}, \widehat{\mathbf{T}})$, and an identification between the $X_*(\mathbf{S}) = X_*(\mathbf{T})_F$ and $X^*(\widehat{\mathbf{S}})$. The inclusion $\mathbf{S} \hookrightarrow \mathbf{T}$ gives an inclusion $X_*(\mathbf{S}) \hookrightarrow X_*(\mathbf{T})$, and this yields a short exact sequence

$$1 \longrightarrow \widehat{\mathbf{T}}^{1-\sigma} \longrightarrow \widehat{\mathbf{T}} \longrightarrow \widehat{\mathbf{S}} \longrightarrow 1 \ ,$$

showing that $\widehat{\mathbf{S}} \simeq \widehat{\mathbf{T}}/(1-\sigma)\widehat{\mathbf{T}}$. Therefore,

$$\widehat{\mathbf{T}} = \mathrm{Spec}(\mathbb{C}[X^*(\widehat{\mathbf{T}})]) = \mathrm{Spec}(\mathbb{C}[X_*(\mathbf{T})]),$$
$$\widehat{\mathbf{S}} = \mathrm{Spec}(\mathbb{C}[X_*(\mathbf{S})]) = \mathrm{Spec}(\mathbb{C}[\Lambda_T]) = \mathrm{Spec}\left(\mathcal{C}_c(\mathbf{T}(F) \mathbin{/\!\!/} \mathcal{T}(\mathcal{O}_F), \mathbb{C})\right).$$

In particular, $\widehat{\mathbf{S}}(\mathbb{C}) = \mathrm{Hom}(X_*(\mathbf{T})_F, \mathbb{C}^\times)$. The above fixed canonical identification $W(\mathbf{G}_{\overline{F}}, \mathbf{T}) \simeq W(\widehat{\mathbf{G}}, \widehat{\mathbf{T}})$, lets $W(\mathbf{G}, \mathbf{S})$ operates on $\widehat{\mathbf{S}}$ by duality. The space $\widehat{\mathbf{S}}/W(\mathbf{G}, \mathbf{S})$ has the structure of a smooth affine $\mathbb{C}$-scheme whose coordinate ring is $\mathbb{C}[X_*(\mathbf{S})]^{W(\mathbf{G},\mathbf{S})}$:

$$\widehat{\mathbf{S}}/W(\mathbf{G}, \mathbf{S}) = \mathrm{Spec}\left(\mathbb{C}[X_*(\mathbf{S})]^{W(\mathbf{G},\mathbf{S})}\right) = \mathrm{Spec}\left(\mathbb{C}[\Lambda_T]^{W(\mathbf{G},\mathbf{S})}\right).$$

Using the twisted Satake isomorphism of Theorem III.10.0.1 we obtain

$$\widehat{\mathbf{S}}/W(\mathbf{G}, \mathbf{S}) = \mathrm{Spec}\left(\mathcal{H}_K(\mathbb{C})\right). \tag{IV.1}$$

## IV.2 Unramified representations and unramified $L$-parameters

Let $\mathcal{W}_F \subset \Gamma_{un}$ whose elements induce an integral power of the Frobenius automorphism $\sigma\colon x \mapsto x^q$ on the algebraic closure of the residue field. The valuation $\mathrm{val}\colon \mathcal{W}_F \to \mathbb{Z}$ sends an element $\psi \in \mathcal{W}_F$ to the power of $\sigma$ it induces, e.g $\mathrm{val}(\sigma) = 1$. Define the "Weyl form" of the Langlands group to be ${}^L_w\mathbf{G} := \widehat{\mathbf{G}} \rtimes \mathcal{W}_F \subset {}^L\mathbf{G}$. The embedding $\mathbb{Z} \to \mathcal{W}_F$ given by $1 \mapsto \sigma$ defines a semidirect product $\widehat{\mathbf{G}} \rtimes \mathbb{Z}$ and we get a homomorphism

$${}^L_w\mathbf{G} \to \widehat{\mathbf{G}} \rtimes \mathbb{Z}.$$

**Definition** IV.2.0.1. *An unramified L-parameter is a homomorphism $\phi\colon \mathcal{W}_F \to {}^L_w\mathbf{G}$ that verifies the following properties:*

1. *The composition $\mathcal{W}_F \xrightarrow{\phi} {}^L_w\mathbf{G} \twoheadrightarrow \mathcal{W}_F$ is the identity.*
2. *For any $w \in \mathcal{W}_F$, $\phi(w)$ is semisimple.*
3. *The composition $\mathcal{W}_F \xrightarrow{\phi} {}^L_w\mathbf{G} \longrightarrow \widehat{\mathbf{G}} \rtimes \mathbb{Z}$ factors through val.*

*Set $\Phi_{un}(\mathbf{G})$ for the set of equivalence[3] classes of unramified L-parameters.*

[3] Two $L$-parameters are equivalent if they are $\widehat{\mathbf{G}}$-conjugate.

The set of $L$-parameters is in bijection with the set of semisimple elements of the form $g \rtimes \sigma \in {}^L\mathbf{G}$. Therefore, $\Phi_{un}(\mathbf{G})$ identifies with the set of semisimple elements of $\widehat{\mathbf{G}}$ modulo $\sigma$-conjugation.

**Definition** IV.2.0.2. *An unramified representation of $\mathbf{G}(F)$ is a homomorphism of groups $\pi\colon \mathbf{G}(F) \to \mathbf{GL}(V)$ where $V$ is a $\mathbb{C}$-vector space verifying the following conditions:*

1. *$\pi$ is irreducible.*
2. *The stabilizer of any vector $v \in V$ is an open subgroups of $\mathbf{G}(F)$.*
3. *For any open subgroup $O \subset \mathbf{G}(F)$, the vector subspace $V^O$ of $O$-fixed vectors is finite dimensional.*
4. *The subspace $V^K$ is nonzero.*

*Set $\Pi_{un}(\mathbf{G})$ for the set of equivalence*[4] *classes of unramified representations of $\mathbf{G}(F)$.*

**Proposition** IV.2.0.1. *There is a natural bijection*

$$\Phi_{un}(\mathbf{G}) \simeq \widehat{\mathbf{S}}(\mathbb{C})/W(\mathbf{G},\mathbf{S}) \simeq \Pi_{un}(\mathbf{G}).$$

*Proof.* In the proof of [BR94, Proposition 1.12.1], one shows first the above proposition for the torus $\mathbf{T}$:

$$\Phi_{un}(\mathbf{T}) \simeq \widehat{\mathbf{S}}(\mathbb{C}) \simeq \Pi_{un}(\mathbf{T}),$$

then deduce it for $\mathbf{G}$ using [Bor79, Proposition 6.7]. □

**Remark** IV.2.0.1. *The above proposition gives an alternative characterization of the twisted Satake homomorphism. Consider the following injective homormophism*

$$\begin{aligned} \mathcal{H}_K(\mathbb{C}) &\longrightarrow \{\Pi_{un}(\mathbf{G}) \to \mathbb{C}\} \\ h_g = \mathbf{1}_{KgK} &\longmapsto (\pi \mapsto \mathrm{Tr}(\pi(h_g)|_{V^K})), \end{aligned}$$

*where, $V$ is given a structure of a left $\mathcal{H}_K(\mathbb{C})$-module defined by $f \cdot v$ for $f \in \mathcal{H}_K(\mathbb{C})$ and $v \in V$ by the formula*

$$f \cdot v = \int_G f(g)(\pi(g)\cdot v)d\mu_K(g).$$

*By Proposition IV.2.0.1 we get the following commutative diagram*

$$\begin{tikzcd} \mathcal{H}_K(\mathbb{C}) \arrow[rr, hook] \arrow[d, "\simeq"] & & \mathcal{C}_c(\mathbf{T}(F) /\!\!/ \mathcal{T}(\mathcal{O}_F), \mathbb{C}) \arrow[d, "\simeq"] \\ \mathbb{C}[\Pi_{un}(\mathbf{G})] \arrow[r, "\simeq"] & \mathbb{C}[\Pi_{un}(\mathbf{T})]^{W(\mathbf{G},\mathbf{S})} \arrow[r, hook] & \mathbb{C}[\Pi_{un}(\mathbf{T})]. \end{tikzcd}$$ □

[4]Two representations $(\pi_1, V_1)$ and $(\pi_2, V_2)$ are equivalent if there exists an isomorphism $V_1 \to V_2$ sending $\pi_1$ to $\pi_2$.

## IV.3 The Hecke polynomial

Let $\mathfrak{c} \in \mathcal{M}(\overline{F})$ and $\mu_{\mathfrak{c}} \in X_*(\mathbf{T})$ be the unique $\mathbf{B}_{\overline{F}}$-dominant cocharacter of $\mathbf{T}_{\overline{F}}$. Both, $\mathfrak{c}$ and $\mu_{\mathfrak{c}}$ have the same field of definition, a finite unramified extension $F(\mathfrak{c}) \subset F^{un}$ of $F$. Set $d = [F(\mathfrak{c}) : F]$ and let

$$\mathrm{Norm}_{F(\mathfrak{c})/F}\, \mathfrak{c} := [\prod_{\tau \in \mathrm{Gal}(F(\mathfrak{c})/F)} \tau(\mu_{\mathfrak{c}})] \in \mathcal{M}(F)$$

be the norm of $\mathfrak{c}$[5]. We may assume that for some representative (and hence for all) of the conjugacy class $\mathrm{Norm}_{F(\mathfrak{c})/F}\, \mathfrak{c}$ takes values in the torus $\mathbf{T}$. The conjugacy class $\mathfrak{c} \in \mathcal{M}(F(\mathfrak{c}))$ determines a Weyl orbit of a character of $\widehat{\mathbf{T}}$, in which there is a unique $\widehat{\mu}_{\mathfrak{c}} \in X^*(\widehat{\mathbf{T}})$ that is dominant with respect to the Borel subgroup $\widehat{\mathbf{B}}$.

Let $(r_{\mathfrak{c}}, V)$ be a representation of ${}^L(\mathbf{G}_{F(\mathfrak{c})})$ (unique up to isomorphism) satisfying the conditions:

- The restriction of $r_{\mathfrak{c}}$ to $\widehat{\mathbf{G}}$ is irreducible with highest weight $\widehat{\mu}_{\mathfrak{c}}$.
- For every admissible invariant splitting of ${}^L(\mathbf{G}_{F(\mathfrak{c})})$ the subgroup $\Gamma_{un}^d$ of ${}^L(\mathbf{G}_{F(\mathfrak{c})})$ acts trivially on the highest weight space of $r_{\mathfrak{c}}$.

Fix an invariant admissible splitting ${}^L(\mathbf{G}_{F(\mathfrak{c})}) = \widehat{\mathbf{G}} \rtimes \Gamma_{un}^d$.

**Definition** IV.3.0.1 (The Hecke polynomial). *For every $\widehat{g} \in \widehat{\mathbf{G}}$, consider the following polynomial*

$$P_{\mathbf{G},\mathfrak{c}}(X) = \det\left(X - q^{d\langle \mu_{\mathfrak{c}}, \rho\rangle} r_{\mathfrak{c}}\left((\widehat{g} \rtimes \sigma)^d\right)\right).$$

*By varying $\widehat{g}$, the coefficients of $P_{\mathbf{G},\mathfrak{c}}$ are viewed as elements of the algebra of regular functions of $\Phi_{un}(\mathbf{G})$.*

Now, combining Proposition IV.2.0.1 and (IV.1) we have

$$\Phi_{un}(\mathbf{G}) \simeq \mathrm{Spec}(\mathcal{H}_K(\mathbb{C})). \tag{IV.2}$$

Accordingly, let $H_{\mathbf{G},\mathfrak{c}} \in \mathcal{H}_K(\mathbb{C})[X]$ be the Hecke polynomial corresponding to $P_{\mathbf{G},\mathfrak{c}}$ via (IV.2) (compare with [BR94, §6]).

[5]It is straightforward that the conjugacy class $\mathrm{Norm}_{F(\mathfrak{c})/F}\, \mathfrak{c}$ does not depend on the choice of the representative $\mu_{\mathfrak{c}}$.

## IV.4 Explicit untwisted Satake transform

Let $\mu \in \mathrm{Norm}_{F(\mathfrak{c})/F}\,\mathfrak{c}$ be the cocharacter of $\mathbf{T}$ which is $\mathbf{B}$-dominant, i.e. $\varpi^{\mu}$ is antidominant. Let $\mathbf{L}$ be the centralizer of $\mu$ in $\mathbf{G}$. Let $\mathbf{P}$ be the largest parabolic subgroup of $\mathbf{G}$ relative to which $\mu$ is dominant, $\mathbf{L}$ is a Levi factor of $\mathbf{P}$ and $\mathbf{U}_P^+$ the unipotent radical of $\mathbf{P}$. By definition we have $\mathbf{T} \subset \mathbf{L}$ and $\mathbf{U}_P^+ \subset \mathbf{U}^+$. Set $K_? = ? \cap K$ for any $? \in \{P, L, U_P^+\}$. Denote by $f_{[\mu]} = \mathbf{1}_{K\varpi^{\mu}K} \in \mathcal{H}_K(R)$ (resp. $g_{[\mu]} = \mathbf{1}_{\varpi^{\mu}K_L} \in \mathcal{C}_c(L /\!\!/ K_L, R)$, resp. $h_{[\mu]} = \mathbf{1}_{\varpi^{\mu}T_1} \in \mathcal{C}_c(T /\!\!/ T_1, R) \simeq R[\Lambda_T]$ the characteristic function of the double coset corresponding to $[\mu]$. Let $p\colon \mathbf{G}_{sc} \to \mathbf{G}$ be the simply connected covering of the derived group of $G$, and let $\mathbf{S}_{sc}$ be the unique maximal $F$-split torus of $\mathbf{G}_{sc}$ such that $p(\mathbf{S}_{sc}) \subset \mathbf{S}$. The map $p$ defines a homomorphism from $X_*(\mathbf{S}_{sc})$ to $X_*(\mathbf{S})$. We are interested in the set

$$\Sigma_F(\mu) = \{\nu \in X_*(\mathbf{S}) : \mu - \nu \in \mathrm{Im}(X_*(\mathbf{S}_{sc})) \text{ and } \mu \geq w\nu \text{ for all } w \in W(\mathbf{G}, \mathbf{S})\}.$$

**Remark** IV.4.0.1. *The above $W$-invariant sets of weights plays a prominent role in representation theory and they are called "saturated sets of weights". Moreover, we have (see [Hum72, 13.4 Exercise] and Bourbaki's [Bou68, Chapter VI, Exercises of §1 and §2]) that*

$$\Sigma_F(\mu) = \bigsqcup_{\lambda \in X_*(\mathbf{S}) \cap \overline{\mathcal{C}}\colon \lambda \preceq \mu} W\lambda$$

*where $\preceq$ denotes the partial order on $X_*(\mathbf{S}) \cap \overline{\mathcal{C}}$ defined by*

$$\lambda \preceq \nu \Leftrightarrow \nu - \lambda = \sum n_\alpha \alpha^\vee, n_\alpha \in \mathbb{Z}_{\geq 0}.$$

We have the following explicit description of the untwisted Satake homomorphism

**Proposition** IV.4.0.1. *Write*

$$\dot{\mathcal{S}}_T^G(f_{[\mu]}) = \sum_{\nu \in \Sigma_F(\mu)} c(\nu).\mathbf{1}_{\varpi^{\nu}T_1} \in \mathcal{C}_c(T /\!\!/ T_1, R),$$

*and the coefficients $\{c(\nu)\}$ are positive powers of $q$ and verifies*

$$c(w\nu) = q^{\langle \delta, \nu - w(\nu)\rangle} c(\nu) \text{ for all } w \in W(\mathbf{G}, \mathbf{S}),$$

*with $c(\mu) = 1$.*

*Proof.* The untwisted Satake isomorphism (Theorem III.10.0.1) ensures that

$$\dot{\mathcal{S}}_T^G(f_{[\mu]}) \in (\Lambda_T \otimes_{\mathbb{Z}} R)^{(W,\bullet)},$$

which shows $c(\nu)q^{\langle \delta, \nu\rangle} = c(w(\mu))q^{\langle \delta, w(\nu)\rangle}$ for all $w \in W(\mathbf{G}, \mathbf{S})$. The fact that $c(\nu) > 0$ if and only $\nu \in \Sigma_F(\mu)$ follows by [Kot84a, Lemma 2.3.7 (a)] for the "only if" and [Rap00] for the "if". Finally, the coefficient $c(\mu) = 1$ is obtained by [Kot84a, Lemma 2.3.7 (b)] using Remark III.10.0.4. $\square$

## IV.5 Seed relations and $\mathbb{U}$-operators

Using the fixed épinglage, we can consider a $\Gamma_{un}$ -equivariant embedding ${}^L\mathbf{T} = \widehat{\mathbf{T}} \rtimes \Gamma_{un} \hookrightarrow {}^L\mathbf{G}$. The composition

$$ {}^L\mathbf{T} \hookrightarrow {}^L\mathbf{G} \xhookrightarrow{r_{\mathfrak{c}}} \mathbf{GL}(V) \xrightarrow{P_{\mathbf{G},\mathfrak{c}}} \mathbb{C}[X] \ , $$

is independent of all fixed choices. The restriction of $r_{\mathfrak{c}}$ to $\widehat{\mathbf{T}}$ yields a weight space decomposition

$$ V = \bigoplus_{\lambda \in \Sigma_E(\mu_{\mathfrak{c}})} V_{\widehat{\lambda}}. $$

We have

$$ \mathcal{S}^G_T(P_{\mathbf{G},\mathfrak{c}}) = \det\left(X - q^{d\langle \mu_{\mathfrak{c}},\rho\rangle} r_{\mathfrak{c}}|_{{}^L(\mathbf{T}_{F(\mathfrak{c})})}\left((\widehat{t} \rtimes \sigma)^d\right)\right) \in \mathbb{C}[\Phi_{un}(\mathbf{T})]^{W(\mathbf{G},\mathbf{S})}. $$

Define the twisted restriction of $r_{\mathfrak{c}}$ to be the morphism of schemes

$$ r_T \colon {}^L(\mathbf{T}_{F(\mathfrak{c})}) = \widehat{\mathbf{T}} \rtimes \Gamma^d_{un} \longrightarrow \mathbf{GL}(V) $$

given on $\mathbb{C}$-points by

$$ r_T(1 \rtimes \sigma^d) = r_{\mathfrak{c}}(1 \rtimes \sigma^d) \text{ and } r_T(\widehat{t} \rtimes 1) \cdot v_\lambda = q^{-\langle \rho,\lambda\rangle} \lambda(\widehat{t}) \cdot v_\lambda \tag{IV.3} $$

for $v_\lambda \in V_\lambda$ for all $\lambda \in \Sigma(\mu_{\mathfrak{c}})$. The homomorphism $r_T$ is not a homomorphism of groups but maps conjugacy classes to conjugacy classes and it is defined to ensure, using Remark III.10.0.4 and (IV.3), that

$$ \begin{aligned} \dot{\mathcal{S}}^G_T(P_{\mathbf{G},\mathfrak{c}}) &= \eta_B \circ \mathcal{S}^G_T(P_{\mathbf{G},\mathfrak{c}}) \\ &= \det\left(X - q^{-d\langle \mu_{\mathfrak{c}},\rho\rangle} r_T\left((\widehat{t} \rtimes \sigma)^d\right)\right) \in \mathbb{C}[\Phi_{un}(\mathbf{T})]. \end{aligned} $$

**Remark** IV.5.0.1. *Note that our choice of the twisted representation $r_T$ depends crucially on the normalization of the isomorphism $X_*(\mathbf{S}) \simeq \Lambda_T$. We have adopted the following isomorphism $\lambda \mapsto \varpi^\lambda$. Using Remark III.10.0.4 and $\delta_B(\varpi^\lambda)^{1/2} = q^{-\langle \lambda,\rho\rangle}$, we see that*

$$ \begin{tikzcd} X_*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{C} \arrow[r, "\eta \colon \lambda \mapsto q^{-\langle \lambda,\rho\rangle}"] \arrow[d, "\simeq"', "\lambda \mapsto \varpi^\lambda T_1"] & X_*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{C} \arrow[d, "\simeq"] \\ \Lambda_T \otimes_{\mathbb{Z}} \mathbb{C} \arrow[r, "\eta \colon tT_1 \mapsto \delta(t)^{1/2} tT_1"] & \Lambda_T \otimes_{\mathbb{Z}} \mathbb{C}. \end{tikzcd} $$

As opposed to [Wed00, Proposition 2.7], we insist on the fact that we do not assume $\mu$ to be minuscule in the following proposition.

**Proposition** IV.5.0.1. *1. Let $\mathbf{S}^{F(\mathfrak{c})} \subset \mathbf{T}$ denotes the maximal split torus of $\mathbf{G}_{F(\mathfrak{c})}$ containing the image of $\mu_{\mathfrak{c}}$, let $\overline{\mathcal{C}}_{F(\mathfrak{c})} \subset \mathcal{B}(\mathbf{G}_{F(\mathfrak{c})}, F(\mathfrak{c}))_{\text{ext}}$ be the closed vectorial chamber corresponding to the Borel $\mathbf{B}_{F(\mathfrak{c})}$. We have*

$$deg(H_{\mathbf{G},\mathfrak{c}}) \geq \sum_{\lambda \in X_*(\mathbf{S}^{F(\mathfrak{c})}) \cap \overline{\mathcal{C}}_{F(\mathfrak{c})} :\, \lambda \preceq \mu_{\mathfrak{c}}} \#(W(\mathbf{G}, \mathbf{S}^{F(\mathfrak{c})})\lambda) = \#\Sigma_{F(\mathfrak{c})}(\mu_{\mathfrak{c}})$$

*2. The twisted restriction $r_T$ of $r_{\mathfrak{c}}$ to ${}^L(\mathbf{T}_{F(\mathfrak{c})})$ is isomorphic to a direct sum*

$$V = \bigoplus_{\Sigma_{F(\mathfrak{c})}(\mu_{\mathfrak{c}})} V_{\widehat{\lambda}}$$

*where $V_{\widehat{\lambda}}$ is one-dimensional with generator $v_{\widehat{\lambda}}$ for any $\widehat{\lambda} \in W(\widehat{\mathbf{G}}, \widehat{\mathbf{T}})\widehat{\mu}$, such that*

$$r_T(\widehat{t} \rtimes \sigma^d) \cdot v_{\sigma^{d(r-1)}(\widehat{\lambda})} = q^{-\langle \rho, \lambda \rangle} \widehat{\lambda}(\widehat{t}) \cdot v_{\widehat{\lambda}}. \tag{IV.4}$$

*Proof.* We will just imitate the proof of [Wed00, (2) Proposition 2.7] but without requiring $\mu$ to be minuscule.

1. Fix a Borel pair $(\widehat{\mathbf{T}}, \widehat{\mathbf{B}})$ of $\widehat{\mathbf{G}}$ and let $\widehat{\mu}_{\mathfrak{c}}$ be the dominant character of $\widehat{\mathbf{T}}$ corresponding to the conjugacy classe $\mathfrak{c}$. By definition of the Hecke polynomial, its degree is the dimension of the representation $r_{\mathfrak{c}}$ which is irreducible with highest weight $\widehat{\mu}_{\mathfrak{c}}$ as a representation of $\widehat{\mathbf{G}}$. By remark IV.4.0.1, the only weights of $r_{\mathfrak{c}}$ are the elements $\bigsqcup_{\widehat{\lambda}} W(\widehat{\mathbf{G}}, \widehat{\mathbf{T}})\widehat{\lambda}$ where the disjoint union is taken over dominant wights $\widehat{\lambda} \preceq \widehat{\mu}_{\mathfrak{c}}$ (here $\preceq$ is the usual partial order on dominant weights $X^*(\widehat{\mathbf{T}})^{\text{dom}}$). By definition of the dual group, we then have
$$\begin{aligned}\bigsqcup_{\widehat{\lambda} \in X^*(\widehat{\mathbf{T}})^{\text{dom}} :\, \widehat{\lambda} \preceq \widehat{\mu}_{\mathfrak{c}}} W(\widehat{\mathbf{G}}, \widehat{\mathbf{T}})\widehat{\lambda} &= \bigsqcup_{\lambda \in X_*(\mathbf{S}^{F(\mathfrak{c})}) \cap \overline{\mathcal{C}}_{F(\mathfrak{c})} :\, \lambda \preceq \mu_{\mathfrak{c}}} W(\mathbf{G}_{F(\mathfrak{c})}, \mathbf{S}^{F(\mathfrak{c})})\lambda \\ &= \Sigma_{F(\mathfrak{c})}(\mu_{\mathfrak{c}}).\end{aligned}$$

2. The twisted restriction $r_T$ of $r_{\mathfrak{c}}$ to ${}^L(\mathbf{T}_{F(\mathfrak{c})})$ is isomorphic to a direct sum
$$V = \bigsqcup_{\widehat{\lambda} \in X^*(\widehat{\mathbf{T}})^{\text{dom}} :\, \widehat{\lambda} \preceq \widehat{\mu}_{\mathfrak{c}}} V_{\widehat{\lambda}}$$
and the highest weight space $V_{\widehat{\mu}_{\mathfrak{c}}}$ is one-dimensional with generator $v_{\widehat{\mu}_{\mathfrak{c}}}$. Accordingly, $V_{\widehat{\lambda}}$ is one-dimensional for any $\widehat{\lambda} \in W(\widehat{\mathbf{G}}, \widehat{\mathbf{T}})\widehat{\mu}_{\mathfrak{c}}$. The conjugacy class $\mathfrak{c}$ being defined over $F(\mathfrak{c})$, we see that $\langle \sigma^n \rangle$ stabilizes $W(\widehat{\mathbf{G}}, \widehat{\mathbf{T}})\widehat{\mu}_{\mathfrak{c}}$. Choose for each classe $Z \in W(\widehat{\mathbf{G}}, \widehat{\mathbf{T}})\widehat{\mu}_{\mathfrak{c}}/\langle \sigma^d \rangle$ a representative $\widehat{\lambda}_Z \in Z$ and a vector $v_{\widehat{\lambda}_Z} \in V_{\widehat{\lambda}_Z}$. Define
$$v_{\sigma^{rd}(\widehat{\lambda}_Z)} := r_{\mathfrak{c}}(1 \rtimes \sigma^{rd}) \cdot v_{\widehat{\lambda}_Z}, \quad \text{for } 1 \leq r < r_Z := \min\{s \colon \sigma^{sd}\widehat{\lambda}_Z = \widehat{\lambda}_Z\}.$$

Therefore, taking $r=-1$ gives

$$\begin{aligned} r_T(\widehat{t}\rtimes\sigma^d)\cdot v_{\sigma^{d(r-1)}(\widehat{\lambda}_Z)} &= r_T(\widehat{t}\rtimes 1)\cdot v_{\widehat{\lambda}_Z} \\ &= q^{-\langle\rho,\lambda\rangle}\widehat{\lambda}_Z(\widehat{t})\cdot v_{\widehat{\lambda}_Z} \qquad \text{by (IV.3).} \ \square \end{aligned}$$

**Lemma** IV.5.0.1. *We have $(\dot{\mathcal{S}}^G_T H_{\mathbf{G},\mathfrak{c}})(\mu)=0$ in $\mathcal{C}_c(T\,/\!\!/\,T_1,R)$.*

*Proof.* The conjugacy classe $[\mu]$ (resp. $\mathfrak{c}$) gave rise to a dominant character $\widehat{\mu}$ (resp. $\widehat{\mu}_{\mathfrak{c}}$) of $\widehat{\mathbf{T}}$ and

$$\widehat{\mu}=\widehat{\mu}_{\mathfrak{c}}\sigma(\widehat{\mu}_{\mathfrak{c}})\cdots\sigma^{d-1}(\widehat{\mu}_{\mathfrak{c}}).$$

To prove the lemma, it suffices to show that

$$\det\left(X-q^{d\langle\mu_{\mathfrak{c}},\rho\rangle}r_T|_{V_{\widehat{\mu}_{\mathfrak{c}}}}\left((\sigma\ltimes\widehat{t})^d\right)\right)\in\mathbb{C}[\Phi_{un}(\mathbf{T})][X]$$

has $\widehat{\mu}(\widehat{t})$ as a root for all $\widehat{t}\in\widehat{\mathbf{T}}$. Identify $\Phi_{un}(\mathbf{T})$ with the set of $\sigma$-conjugacy classes $\{\widehat{t}\}$ of elements $\widehat{t}\in\widehat{\mathbf{T}}(\mathbb{C})$. For any $v\in V_{\widehat{\mu}}$, we have

$$\begin{aligned} q^{d\langle\mu_{\mathfrak{c}},\rho\rangle}r_T\left((\sigma\ltimes\widehat{t})^d\right)\cdot v &= q^{d\langle\mu_{\mathfrak{c}},\rho\rangle}r_T\left(\sigma^d\ltimes(\widehat{t}\sigma(\widehat{t})\cdots\sigma^{d-1}(\widehat{t}))\right)\cdot v \\ &\overset{\text{Prop. IV.5.0.1}}{=} \widehat{\mu}_{\mathfrak{c}}\left(\widehat{t}\sigma(\widehat{t})\cdots\sigma^{d-1}(\widehat{t})\right)\cdot v \\ &= \widehat{\mu}_{\mathfrak{c}}(\widehat{t})\sigma(\widehat{\mu}_{\mathfrak{c}})(\widehat{t})\cdots\sigma^{d-1}(\widehat{\mu}_{\mathfrak{c}})(\widehat{t})\cdot v \\ &= \widehat{\mu}(\widehat{t})\cdot v. \quad\square \end{aligned}$$

We will show now the following "seed relation"

**Theorem** IV.5.0.1 (Seed relation). *The operator $\dot{\Theta}_\mu=i_{\varpi^\mu}\in\mathbb{U}$ is a right root of the Hecke polynomial $H_{\mathbf{G},\mathfrak{c}}$ in the non-commutatif $R$-algebra* $\mathrm{End}_P(\mathcal{C}_c(G/K,R))$.

*Proof.* First of all, since $\mu$ is dominant with respect to $\mathbf{B}$, we have $\varpi^\mu\in T^-$, thus $\dot{\Theta}_\mu\in\mathbb{U}$. Under the identifications $\Lambda_T\simeq X_*(\mathbf{T})_F\simeq X^*(\widehat{\mathbf{T}})_F$ the element $\varpi^{\mu^{-1}}T_1\in\Lambda_T^-$ corresponds to the function $t\mapsto\widehat{\mu}(t)$.

Recall that by Lemma III.14.2.3 $\dot{\Theta}_\mu\in\mathrm{End}_P\mathcal{C}_c(G\,/\!\!/\,K,\mathbb{Z})$ and the coefficients of $H_{\mathbf{G},\mathfrak{c}}$ are in $\mathcal{H}_K(R)\simeq\mathrm{End}_G\mathcal{C}_c(G\,/\!\!/\,K,R)$, thus

$$H_{\mathbf{G},\mathfrak{c}}(\dot{\Theta}_\mu)\in\mathrm{End}_P\mathcal{C}_c(G\,/\!\!/\,K,R).$$

Using Theorem III.12.0.1, we see that $\dot{\Theta}_{\mathrm{Bern}}\circ\dot{\mathcal{S}}^G_T H_{\mathbf{G},\mathfrak{c}}\in Z(\mathcal{H}_I(R))[X]$. Write $H_{\mathbf{G},\mathfrak{c}}=\sum_{k=1}^r h_k X^k$, and $\bar{h}_k=\dot{\Theta}_{\mathrm{Bern}}\circ\dot{\mathcal{S}}^G_T(h_k)\in Z(\mathcal{H}_I(R))$. So $\bar{h}_k *_I \mathbf{1}_K=\mathbf{1}_K *_I \bar{h}_k=h_K$. We then have for any $p \in P$

$$\begin{aligned}
\mathbf{1}_{pK} \bullet H_{\mathbf{G},\mathfrak{c}}(\dot{\Theta}_\mu) &= \sum_{k=1}^{r} (\mathbf{1}_{pK} \bullet \dot{\Theta}_\mu^k) *_K h_k \\
&= \sum_{k=1}^{r} (\mathbf{1}_{pI} *_I \dot{\Theta}_\mu^k) *_I \mathbf{1}_K *_K h_k \\
&= \sum_{k=1}^{r} (\mathbf{1}_{pI} *_I \dot{\Theta}_\mu^k) *_I (\frac{1}{[K:I]} \mathbf{1}_K *_I \mathbf{1}_K *_I \bar{h}_k) \\
&= \sum_{k=1}^{r} (\mathbf{1}_{pI} *_I \dot{\Theta}_\mu^k) *_I \mathbf{1}_K *_I \bar{h}_k \\
&= \mathbf{1}_{pI} *_I \left( \sum_{k=1}^{r} \dot{\Theta}_\mu^k *_I \bar{h}_k \right) *_I \mathbf{1}_K \\
&= \mathbf{1}_{pI} *_I \left( \sum_{k=1}^{r} \bar{h}_k *_I \dot{\Theta}_\mu^k \right) *_I \mathbf{1}_K \\
&= \mathbf{1}_{pI} *_I \left( (\dot{\Theta}_{\mathrm{Bern}} \circ \dot{\mathcal{S}}_T^G H_{\mathbf{G},\mathfrak{c}})(\dot{\Theta}_\mu) \right) *_I \mathbf{1}_K \\
&= \mathbf{1}_{pI} *_I \dot{\Theta}_{\mathrm{Bern}} \left( (\dot{\mathcal{S}}_T^G H_{\mathbf{G},\mathfrak{c}})(\varpi^\mu T_1) \right) *_I \mathbf{1}_K \\
&\overset{\text{Lemma IV.5.0.1}}{=} 0.
\end{aligned}$$

We have shown

$$H_{\mathbf{G},\mathfrak{c}}(\dot{\Theta}_\mu) = \sum_{k=1}^{r} h_k \circ \dot{\Theta}_\mu^k = 0 \in \mathrm{End}_P(\mathcal{C}_c(G/K, R)). \quad \square$$

**Remark** IV.5.0.2. *If $\mu_\mathfrak{c}$ is minuscule, then $\Sigma_F(\mu_\mathfrak{c}) = W(\mathbf{G}_{\overline{F}}, \mathbf{T})\,\mu_\mathfrak{c}$ and accordingly the degree of the Hecke polynomial is*

$$\deg(H_{\mathbf{G},\mathfrak{c}}) = |W(\mathbf{G}_{\overline{F}}, \mathbf{T})\,\mu_\mathfrak{c}|\,.$$

*In particular, $\deg(H_{\mathbf{G},\mathfrak{c}}) \geq \deg(P_\mu) = |W/W_\mu| = |W(\mathbf{G}, \mathbf{S})\,\mu|$, where $P_\mu$ is the minimal polynomial of $\dot{\Theta}_\mu$ in $Z(\mathcal{H}_I(R))$ (see proof of Theorem III.13.0.1). Therefore, if $\mathbf{G}$ is a split group, $\mu_\mathfrak{c}$ minuscule and $E = F$, then*

$$H_{G,[\mu]} = P_\mu *_I \mathbf{1}_K. \quad \square$$

## IV.6 Bültel's annihilation relation

In this last section we will show how Theorem IV.5.0.1 lifts (generalizes) a previously known result due to Bültel [Bül97, 1.2.11].

Let $\dot{\mathcal{S}}_P\colon \mathcal{C}_c(P/K_P,\mathbb{Q}) \to \mathcal{C}_c(L/K_L,\mathbb{Q})$ be the canonical homomorphism given by

$$f \mapsto \left( m \mapsto \int_{U_P^+} f(nm) d\mu_{U_P^+}(n) \right),$$

where $d\mu_{U_P^+}$ is the left-invariant Haar measure giving $K_{U_P^+}$ volume 1. Both $\mathbb{Q}$-modules $\mathcal{C}_c(P/K_P,\mathbb{Q})$ and $\mathcal{C}_c(L/K_L,\mathbb{Q})$ are actually $\mathbb{Q}$-algebras (by Lemma III.5.0.1) and the transform $\dot{\mathcal{S}}_P$ is an algebra homomorphism. Indeed, let $f, g \in \mathcal{C}_c(P/K_P,\mathbb{Q})$ then

$$\begin{aligned}
\dot{\mathcal{S}}_P(f *_{K_P} g)(p) &= \int_{U_P^+} (\int_P f(a) g(a^{-1}up) d\mu_P(a)) d\mu_{U_P^+}(u) \\
&= \int_{U_P^+} \int_L \int_{U_P^+} f(nm) g(m^{-1}n^{-1}up) d\mu_{U_P^+}(n) d\mu_L(m) d\mu_{U_P^+}(u) \\
&= \int_{U_P^+} \left( \int_L f(nm) d\mu_{U_P^+}(n) \right) \left( \int_{U_P^+} g(m^{-1}pu) d\mu_{U_P^+}(u) \right) d\mu_L(m) \\
&= \dot{\mathcal{S}}_P(f) *_{K_P} \dot{\mathcal{S}}_P(g)(p)
\end{aligned}$$

where, $d\mu_P$ denotes the left invariant Haar measure giving $K_P$ measure 1.

We also consider the map $|_P$ sending any function on $G$ to its restriction to $P$. Using the Iwasawa decomposition $G = PK$ (Proposition III.7.0.3) one shows that this is actually an algebra homomorphism

$$|_P\colon \mathcal{H}_K(R) \longrightarrow \mathcal{C}_c(P \mathbin{/\!\!/} K_P, R),$$

and a $|_P$-linear module homomorphism

$$|_P\colon \mathcal{C}_c(G/K,R) \longrightarrow \mathcal{C}_c(P/K_P,R).$$

Using the Iwasawa decomposition $G = PK$ (Proposition III.7.0.3).

**Lemma** IV.6.0.1. *Let $p \in P$ and $m \in L$, then:*

$$\mathbf{1}_{pK}|_P = \mathbf{1}_{pK_P} \text{ and } \dot{\mathcal{S}}_L^P(\mathbf{1}_{mK_P}) = |mK_{U_P^+}m^{-1}|_{U_P^+} \mathbf{1}_{mK_L}.$$

*Proof.* The first equality is a direct consequence of the Iwasawa decomposition. For the second it is deduced from the fact that $K_P = K_L.K_{U_P^+}$ given in Proposition III.7.0.3:

$$\dot{\mathcal{S}}_L^P(\mathbf{1}_{mK_P})(a) = \int_{U_P^+} \mathbf{1}_{mK_P}(ua) d\mu_{U_P^+}(u).$$

The integrand is nonzero if and only if $ua \in mK_P = mK_L \cdot K_{U_P^+}$, but since $L \cap U_P^+ = \{1\}$, we have

$$u \in aK_{U_P^+}a^{-1} \text{ and } a \in mK_L,$$

which is equivalent to $u \in mK_{U_P^+}m^{-1}$ and $w \in mK_L$. Therefore,

$$\dot{\mathcal{S}}_L^P(\mathbf{1}_{mK_P}) = |mK_{U_P^+}m^{-1}|_{U_P^+} \mathbf{1}_{mK_L}. \quad \square$$

Observe that if $mK_{U_P^+}m^{-1} \subset K_{U_P^+}$ then

$$|mK_{U_P^+}m^{-1}|_{U_P^+} = \frac{1}{[K_{U_P^+} : mK_{U_P^+}m^{-1}]} = \frac{1}{[K_P : mK_Pm^{-1}]}.$$

**Lemma** IV.6.0.2. *We have a following commutative diagram of $R$-algebras*

$$\begin{tikzcd} \mathcal{H}_K(R) \arrow[r, hook, "\dot{\mathcal{S}}_L^G"] \arrow[d, "\dot{\mathcal{S}}_T^G"', "\simeq"] & \mathcal{C}_c(L /\!\!/ K_L, R) \arrow[d, "\simeq"', "\dot{\mathcal{S}}_T^L"] \\ R[\Lambda_T]^{(W,\bullet)} \arrow[r, hook] & R[\Lambda_T]^{(W_L,\bullet)} \end{tikzcd}$$

*where, $W_L$ denotes the relative Weyl group of $L$ (which is equal to the subgroup $W_\mu$ of elements in $W$ fixing $\mu$). The lowest horizontal arrow is the inclusion of $W$-invariants into $W_L$-invariants.*

*Proof.* By definition of the parabolic $P$, multiplication in $G$ gives a bijection

$$(U^+ \cap L) \cdot U_P^+ \xrightarrow{\sim} U^+. \qquad \text{(IV.5)}$$

For any $m \in L$ and $h \in \mathcal{H}_K(R)$

$$\begin{aligned} \dot{\mathcal{S}}_T^G(h)(m) &= \int_{U^+} h(um)d\mu_{U^+}(u) && \text{Lemma III.10.0.2} \\ &= \int_{U_P^+}\int_{U^+\cap L} h(u_1u_2m)d\mu_{U_P^+}(u_1)\, d\mu_{U^+\cap L}(u_2) && \text{by (IV.5)} \\ &= \int_{U^+\cap L}\left(\int_{U_P^+} h(u_1u_2m)d\mu_{U_P^+}(u_1)\right) d\mu_{U^+\cap L}(u_2) \\ &= \int_{U^+\cap L} \dot{\mathcal{S}}_L^G(h)(u_2m)d\mu_{U^+\cap L}(u_2) \\ &= \dot{\mathcal{S}}_T^L \circ \dot{\mathcal{S}}_L^G(h)(m), \end{aligned}$$

Therefore, $\dot{\mathcal{S}}_T^G = \dot{\mathcal{S}}_T^L \circ \dot{\mathcal{S}}_L^G$ which confirm the claimed commutativity of the above diagram. Finally, the vertical maps are isomorphisms by Theorem III.10.0.1. □

Let us reformulate the above untwisted Satake homomorphism $\dot{\mathcal{S}}_L^G$ as a homomorphism of endomorphism rings. We have a commutative diagram:

$$\begin{tikzcd} \mathcal{H}_K(R) \arrow[r, "|_P"] \arrow[d, equal] & \mathcal{C}_c(P /\!\!/ K_P, R) \arrow[rr, hook, "\dot{s}_P"] \arrow[d, equal] & & \mathcal{C}_c(L /\!\!/ K_L, R) \arrow[d, equal] \\ \mathrm{End}_G\mathcal{C}_c(G/K,R) \arrow[r, hook, "(1)"] & \mathrm{End}_P\mathcal{C}_c(G/K,R) \arrow[r, "(2)"] & \mathrm{End}_L\mathcal{C}_c(L/K_L,R) \arrow[r, "(3)", "\simeq"'] & \mathrm{End}_L\mathcal{C}_c(L/K_L,R). \end{tikzcd}$$

Let us first say few words about the homomorphisms (1) and (2):

(1) We have used the Iwasawa decomposition $G = PK$ to identify $G/K \simeq P/K_P$ for the middle vertical arrow, accordinly the homomorphism $|_P$ induces the canonical

injection (1):

$$\mathrm{End}_G\mathcal{C}_c(G/K,R) \hookrightarrow \mathrm{End}_P\mathcal{C}_c(G/K,R).$$

(2) We have a homomorphism of rings

$$\mathrm{End}_P\mathcal{C}_c(G/K,R) \longrightarrow \mathrm{End}_P\mathcal{C}_c(U_P^+\backslash G/K,R)$$

$$f \longmapsto (U_P^+gK \mapsto \Pi(f(gK)))$$

where $\Pi$ is the natural obvious map $R[G/K] \to R[U_P^+\backslash G/K]$. But since $P = LU_P^+$, we actually have $\mathrm{End}_P\mathcal{C}_c(U_P^+\backslash G/K,R) = \mathrm{End}_L\mathcal{C}_c(U_P^+\backslash G/K,R)$.

Using the Iwasawa decomposition again $G = U_P^+LK$, we get a bijection

$$U_P^+\backslash G/K \simeq L/K_L.$$

Thus, the homomorphism (2) is the composition

$$\mathrm{End}_P\mathcal{C}_c(G/K,R) \longrightarrow \mathrm{End}_L\mathcal{C}_c(U_P^+\backslash G/K,R) \xrightarrow{\simeq} \mathrm{End}_L\mathcal{C}_c(L/K_L,R).$$

(3) The homomorphism (3) is the twist by the modulus function $\delta$.

**Lemma** IV.6.0.3. *The operator $\dot{\Theta}_\mu = i_{\varpi^\mu}$ lives in $\mathrm{End}_P\mathcal{C}_c(G/K,R)$ and its image by the composition* $(3)\circ(2)$ *is precisely* $g_{[\mu]}$.

*Proof.* Let us first compute the image of the operator $\dot{\Theta}_\mu = i_{\varpi^\mu}$ by the map (2). We have for all $a \in L$ (see Lemma III.14.2.3)

$$\begin{aligned}\dot{\Theta}_\mu(\mathbf{1}_{U_P^+aK}) &= \sum_{p' \in [U_P^+\cap I^+/U_P^+\cap \varpi^\mu I^+\varpi^{-\mu}]} \mathbf{1}_{U_P^+ap'\varpi^\mu K} \\ &= \#(U_P^+\cap I^+/U_P^+\cap \varpi^\mu I^+\varpi^{-\mu})\mathbf{1}_{U_P^+a\varpi^\mu K}\end{aligned}$$

Hence (by Lemma V.2.3.2) the image of $\dot{\Theta}_\mu \in \mathrm{End}_P\mathcal{C}_c(G/K,R)$ by (2) is

$$\#(I^+/\varpi^\mu I^+\varpi^{-\mu})g_{[\mu]} = \delta_B(\varpi^{-\mu})g_{[\mu]} = q^{2\langle\mu,\rho\rangle}g_{[\mu]}.$$

Finally, upon applying (3) shows that the image of $\dot{\Theta}_\mu$ by the composition $(3)\circ(2)$ is $g_{[\mu]} \in \mathrm{End}_L\mathcal{C}_c(L/K_L,R)$. □

Bültel's annihilation result we have mentioned earlier is:

**Corollary** IV.6.0.1 (Bültel's annihilation). *We have*

$$\dot{\mathcal{S}}_L^G(H_{\mathbf{G},\mathfrak{c}}(g_{[\mu]})) = 0 \in \mathcal{C}_c(L /\!\!/ K_L,R).$$

Bültel's result as stated in [Wed00, §2.9] requires the conjugacy class $\mathfrak{c}$ to be minuscule. We will derive this corollary from Theorem IV.5.0.1, showing that the assumption "minuscule" is superfluous.

*Proof.* By definition of the "excursion" pairing §III.14.2 and the proof of Lemma IV.6.0.3, we see that for all $p \in P$:

$$\begin{aligned} 0 &\overset{\text{Theorem IV.5.0.1}}{=} (H_{\mathbf{G},\mathfrak{c}}(\Theta_\mu) \bullet \mathbf{1}_{pK})\,|_P \\ &= \mathbf{1}_{pK_P} *_{K_P} \mathbf{1}_{K_P\varpi^\mu K_P} *_{K_P} (H_{\mathbf{G},\mathfrak{c}})|_P. \end{aligned}$$

This shows that

$$(H_{\mathbf{G},\mathfrak{c}})|_P(\mathbf{1}_{K_P\varpi^\mu K_P}) = 0,$$

and consequently we conclude

$$\begin{aligned} \dot{\mathcal{S}}_G^L(H_{\mathbf{G},\mathfrak{c}})(g_{[\mu]}) &= \dot{\mathcal{S}}_P\left((H_{\mathbf{G},\mathfrak{c}})|_P(\mathbf{1}_{K_P\varpi^\mu K_P})\right) \\ &= 0. \quad \square \end{aligned}$$

# CHAPTER V

# GEOMETRIC REALIZATION OF $\mathbb{U}$-OPERATORS

The goal of this chapter is to translate the purely group theoretic $\mathbb{U}$ operators into a more combinatorial fashion, by describing their induced action on the extended building. This will provide a new class of geometric operators on the set of special vertices justifying why $\mathbb{U}$-operators may be thought of as a conceptual generalization of the successor operators for trees with a marked end.

We continue with the notation adopted in §II.3 and §III: We have previously associated to the fixed maximal split torus $\mathbf{S}$ an apartment $\mathcal{A}$ in the reduced building $\mathcal{B}_{\text{red}}$ and an apartment $\mathcal{A}_{\text{ext}} = \mathcal{A} \times V_G$ in the extended building $\mathcal{B}_{\text{ext}}$ (§II.3.7). We have also fixed an alcove $\mathfrak{a} \subset \mathcal{A}$, and a special point $a_\circ \in \overline{\mathfrak{a}}$. Recall that $K$ denotes the special maximal parahoric subgroup associated to the special point $a_\circ$, and $\widetilde{K}$ its fixator (pointwise stabilizer) in $G^1$. Likewise, $I$ denotes the Iwahori subgroup associated to the alcove $\mathfrak{a}$, and $\tilde{I}$ its fixator (pointwise stabilizer) in $G^1$.

Since we will be manipulating objects in the reduced and extended buildings, we will use subscript $\square_{\text{red}}$ and $\square_{\text{ext}}$ to indicate in which situation we are.

Likewise, to every maximal split torus $\mathbf{S}'$ of $\mathbf{G}$ is naturally attached an (extended) affine apartment $\mathcal{A}_{\text{ext}}(\mathbf{S}') = \mathcal{A}_{\text{red}}(\mathbf{S}') \times V_G$, endowed with an action of $\mathbf{N}_\mathbf{G}(\mathbf{S}')$ such that $\mathbf{Z}_\mathbf{G}(\mathbf{S}')$ yields precisely the set of affine translations of $\mathcal{A}_{\text{ext}}(\mathbf{S}')$. Set $W(\mathbf{S}')$ for the relative Weyl group $\mathbf{N}_\mathbf{G}(\mathbf{S}')/\,\mathbf{Z}_\mathbf{G}(\mathbf{S}')$. Now, since the set of maximal split tori is in bijection with the set of affine apartments, we will omit indicating the corresponding torus, and only write $\mathcal{A}'_{\text{red}}$, $\mathcal{A}'_{\text{ext}}$ and $W'$.

## V.1 Retractions and $\mathcal{U}$-operators

### V.1.1 Preliminaries

**Definition** V.1.1.1. *An element $g \in G$ is said to be strongly type-preserving if: For each triplet $(\mathcal{F}, \mathcal{A}'_{\text{ext}}, w)$, where $\mathcal{F}$ is a facet contained in an apartment $\mathcal{A}'_{\text{ext}}$ and $w \in W'_{\text{aff}}$, for which*

$$\text{(i) } g \cdot \mathcal{F} \subset \mathcal{A}'_{\text{ext}} \text{ and (ii) } (w \circ g) \cdot \mathcal{F} = \mathcal{F},$$

*then the element $w \circ g$ fixes pointwise the closure $\overline{\mathcal{F}}$. Set $G_{tp}$ for the set of strongly type-preserving elements in $G$. A subgroup $H \subset G_{tp}$ of strongly type-preserving elements* *is said to be strongly transitive if it acts transitively on all pairs* $(\mathfrak{a}' \times V_G, \mathcal{A}'_{\text{ext}})$*, where* $\mathcal{A}'_{\text{ext}} \subset \mathcal{B}_{\text{ext}}$ *is an apartment and* $\mathfrak{a}' \times V_G \subset \mathcal{A}'_{\text{ext}}$ *is an alcove.*

For more on these types of automorphisms on buildings we refer to [Rou09].

**Proposition** V.1.1.1. *The subgroup* $G_{tp}$ *of strongly type preserving elements of* $\mathcal{B}_{\text{ext}}$ *is equal to* $G_1 = \ker \kappa_G$*, and its action on* $\mathcal{B}_{\text{ext}}$ *is also strongly transitive.*

*Proof.* According to [Rou09, 11.10], the subgroup of $G$ consisting of strongly type preserving automorphisms of $\mathcal{B}_{\text{ext}}$ is the group generated by $\nu_{N,\text{ext}}^{-1}(W_{\text{aff}})$ [1] and the root groups $U_\alpha$ for $\alpha \in \Phi$. This is equal to the subgroup $G^{\text{aff}}$ generated by parahoric subgroups of $G$ [Vig16, §3.9], but by [Ric16, Lemma 1.3] there is an equality $G^{\text{aff}} = G_1$. Finally, $G_1$ is indeed strongly transitive on $\mathcal{B}_{\text{ext}}$ as shown in [BT72, Corollaire 2.2.6]. □

**Remark** V.1.1.1. *The action of* $G$ *on* $\mathcal{B}_{\text{ext}}$ *is strongly transitive but non type-preserving in general. One may think of* $G_1 \backslash \mathcal{B}_{\text{ext}}$ *as the "universal closed alcove" on which* $G/G_1$ *acts faithfully: it is a (commutative) group of automorphisms of the universal closed alcove, with a translation part, given by* $\nu_G$*, and a "rotational part", finite, given by the torsion subgroup* $G^1/G_1$*.*

### V.1.2 Iwahori subgroups and retractions

For any minimal facet $\{a\} \times V_G \in \mathcal{B}_{\text{ext}}$, there is an apartment $\mathcal{A}_{\text{ext},a} \subset \mathcal{B}_{\text{ext}}$ containing $\mathfrak{a} \times V_G$ and $\{a\} \times V_G$ (see Proposition II.3.6.1 (4)).

**Lemma** V.1.2.1. *For any point* $a \in \mathcal{B}_{\text{ext}}$ *there is an apartment* $\mathcal{A}_{\text{ext},a}$ *containing* $\{a\} \times V_G$ *and* $a$ *(along, then, with the whole facet of* $a$*). (i) There is a unique isometry* $\phi\colon \mathcal{A}_{\text{ext},a} \to \mathcal{A}_{\text{ext}}$ *fixing* $\overline{\mathfrak{a}} \times V_G$*. (ii) The image* $\phi(a)$ *does not depend on the choice of* $\mathcal{A}_{\text{ext},a}$*. Denote this image by* $r_{\mathcal{A}_{\text{ext}},\mathfrak{a}}(a)$*.*

*Proof.* (i) (Existence) By Proposition V.1.1.1, there exists $g \in G$ sending the pair $(\mathcal{A}_{\text{ext},a}, \mathfrak{a} \times V_G)$ to $(\mathcal{A}_{\text{ext}}, \mathfrak{a} \times V_G)$ and fixing pointwise $\overline{\mathfrak{a}} \times V_G$.

(Uniqueness) If $\phi\colon \mathcal{A}_{\text{ext},a} \to \mathcal{A}_{\text{ext}}$ is a second isomorphism, then $\phi \circ g^{-1}$ is an isometric automorphism of $\mathcal{A}_{\text{ext}}$ fixing $\overline{\mathfrak{a}} \times V_G$, but any automorphism of $\mathcal{B}_{\text{ext}}$ that fixes a

[1] The map $\nu_{N,\text{ext}}$ is defined above Remark II.3.7.1.

chamber is the identity. Moreover, being an isometry, $\phi$ fixes $\overline{\mathfrak{a}} \times V_G$ if and only if it fixes the whole intersection $\mathcal{A}_{\mathrm{ext}} \cap \mathcal{A}_{\mathrm{ext},a}$.

(ii) Let $\mathcal{A}'_{\mathrm{ext},a}$ be another apartment containing $\mathfrak{a} \times V_G$ and $a$. Consider the following (possibly non-commutative) diagram

$$\begin{tikzcd} \mathcal{A}_{\mathrm{ext},a} \arrow[r, "\phi"] \arrow[dr, "\phi''"'] & \mathcal{A}_{\mathrm{ext}} \\ & \mathcal{A}'_{\mathrm{ext},a} \arrow[u, "\phi'"'] \end{tikzcd}$$

where all maps are (unique) isomorphism fixing $\overline{\mathfrak{a}} \times V_G$. The uniqueness of (i) implies $\phi = \phi' \circ \phi''$, which shows that the above diagram is commutative. In addition, the map $\phi''$ being an isometry, must fix $a \in \mathcal{A}_{\mathrm{ext},a} \cap \mathcal{A}'_{\mathrm{ext},a}$ for all $v \in V_G$. Hence:

$$\phi(a) = \phi' \circ \phi''(a) = \phi'(a).$$

This proves that $\phi(a)$ does not depend on the chosen $\mathcal{A}_{\mathrm{ext},a}$. □

**Remark** V.1.2.1. *By the above lemma, for any point* $(a \times v) \subset \mathcal{A} \times V_G = \mathcal{A}_{\mathrm{ext}}$ *contained in some apartment* $\mathcal{A}_{\mathrm{ext},a}$ *that also contains* $\mathfrak{a} \times V_G$*, there is a unique isomorphism* $\phi\colon \mathcal{A}_{\mathrm{ext},a} \to \mathcal{A}_{\mathrm{ext}}$ *fixing* $\overline{\mathfrak{a}} \times V_G$*. In particular* $\phi$ *fixes the component* $V_G$*, thus*

$$r_{\mathcal{A}_{\mathrm{ext}},\mathfrak{a}}((a,v)) = (r_{\mathcal{A},\mathfrak{a}}(a), v),$$

*for some retraction map* $r_{\mathcal{A},\mathfrak{a}}\colon \mathcal{B}_{\mathrm{red}} \to \mathcal{A}$ *for the reduced building.*

**Definition** V.1.2.1. *The map* $r_{\mathcal{A}_{\mathrm{ext}},\mathfrak{a}}\colon \mathcal{B}_{\mathrm{ext}} \to \mathcal{A}_{\mathrm{ext}}$ *(resp.* $r_{\mathcal{A},\mathfrak{a}}$*) defined by lemma V.1.2.1 is called the retraction onto* $\mathcal{A}_{\mathrm{ext}}$ *based at* $\mathfrak{a} \times V_G$*, (resp.* $\mathcal{A}$ *based at* $\mathfrak{a}$*).*

In the following lemma, we interpret the retraction $r_{\mathcal{A}_{\mathrm{ext}},\mathfrak{a}}$ using the Iwahori subgroup.

**Lemma** V.1.2.2. *For any* $(a, v) \in \mathcal{A}_{\mathrm{ext}}$*,*

$$r^{-1}_{\mathcal{A}_{\mathrm{ext}},\mathfrak{a}}((a,v)) = \tilde{I} \cdot (a,v) = I \cdot (a,v),$$

*The fibers of the retraction* $r_{\mathcal{A}_{\mathrm{ext}},\mathfrak{a}}$ *are exactly the* $\tilde{I}$*-orbits on* $\mathcal{B}_{\mathrm{ext}}$*.*

**Remark** V.1.2.2. *Using Remark V.1.2.1, we see that for any* $(a, v) \in \mathcal{A}_{\mathrm{ext}}$ *where* $a \in \mathcal{A}$ *and* $v \in V_G$*, we have*

$$r^{-1}_{\mathcal{A},\mathfrak{a}}(a) = I \cdot a = \tilde{I} \cdot a,$$

*since* $I \subset \tilde{I} \subset G^1 = \ker \nu_G$*.*

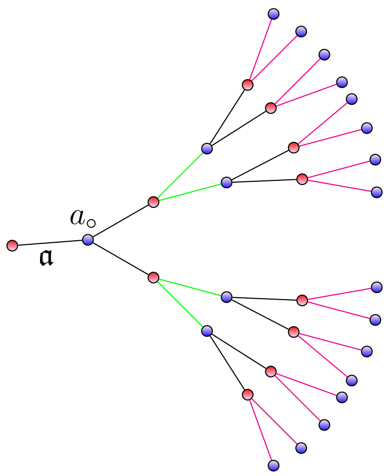
*Proof.* Let $a$ be any point in the apartment $\mathcal{A}$, $a' \in \mathcal{B}_{\mathrm{red}}$ such that $r_{\mathcal{A},\mathfrak{a}}(a') = a$. According to Proposition V.1.1.1, the subgroup $G_1$ has a strongly transitive action on $\mathcal{B}_{\mathrm{ext}}$, thus also on $\mathcal{B}_{\mathrm{red}}$. Hence there exists $g \in G_1$ that sends the ordered pair $(a', \mathfrak{a})$ to the pair $(a, \mathfrak{a})$. In particular $g \cdot \mathfrak{a} = \mathfrak{a}$, but since $G_1$ is strongly type-preserving it must fixe pointwise $\mathfrak{a}$, hence $g \in I$.

The isomorphism $\phi_{g^{-1}} \colon g \cdot \mathcal{A}_{\mathrm{ext}} \to \mathcal{A}_{\mathrm{ext}}$ fixes the alcove $\mathfrak{a} \times V_G$. Thus, by lemma V.1.2.1, we have

$$(a, v) = (\phi_{g^{-1}}(a'), v) = r_{\mathcal{A}_{\mathrm{ext}},\mathfrak{a}}((a', v)), \quad \forall v \in V_G.$$

Conversely, any $(a', v) = g \cdot (a, v)$ clearly retracts to $(r_{\mathcal{A},\mathfrak{a}}(a), v) = r_{\mathcal{A}_{\mathrm{ext}},\mathfrak{a}}(a, v)$. Thus $r_{\mathcal{A},\mathfrak{a}}^{-1}(a) = I \cdot a$, and $r_{\mathcal{A}_{\mathrm{ext}},\mathfrak{a}}^{-1}(a, v) = I \cdot (a, v)$, for any $v \in V_G$. Finally, the equality $I \cdot (a, v) = \tilde{I} \cdot (a, v)$ follows from $\tilde{I} = M^1 I$ (see §III.15.1). $\square$

The following figure describes the case of a tree (e.g., the $\mathbf{SL}_2$ or $\mathbf{U}(3)$ case). The blue vertices in the closure of a green (resp. a magenta) alcove are all in the same $I$-orbit.




### V.1.3 Geometric $\mathbb{U}$-operators

Set

$$\mathcal{A}_{\mathrm{ext}}^{\circ} := M \cdot (a_\circ, 0) = \{a_m := (a_\circ + \nu_M(m), \nu_G(m)) : \forall m \in M\} \subset \mathcal{A}_{\mathrm{ext}}.$$

We define an "excursion pairing"

$$M \times \mathbb{Z}[\mathcal{A}_{\mathrm{ext}}^{\circ}] \longrightarrow \mathbb{Z}[\mathcal{B}_{\mathrm{ext}}^{\circ}], \qquad (m, a_{m'}) \longmapsto \mathcal{U}_m a_{m'}$$

where $\mathcal{U}_m a_{m'}$ is defined to be the formal sum of vertices appearing in the fiber

$$r_{\mathcal{A}_{\mathrm{ext}},m'\cdot\mathfrak{a}}^{-1}(m \cdot a_{m'}) = r_{\mathcal{A}_{\mathrm{ext}},m'\cdot\mathfrak{a}}^{-1}(a_{mm'}).$$

We will see below how the restriction of the above pairing to the semigroup $M^-$ defines a left action $M^- \times \mathbb{Z}[\mathcal{A}^\circ_{\text{ext}}] \longrightarrow \mathbb{Z}[\mathcal{B}^\circ_{\text{ext}}]$. By Lemma V.1.2.2, we have

$$\begin{aligned}
\mathcal{U}_m a_{m'} &= I_{m'\cdot\mathfrak{a}} \cdot a_{mm'} \\
&= m' I m'^{-1} m m' \cdot (a_\circ, 0) \\
&\overset{(1)}{=} m' I m \cdot (a_\circ, v) \\
&\overset{(2)}{=} m' I^+ m \cdot (a_\circ, 0) \qquad\qquad\qquad\qquad \text{(Lemma III.7.0.3)},
\end{aligned}$$

For (1), although $m, m'$ may not commute as elements of $M$, they do commute modulo $M_1 \subset I \subset \tilde{I}$ which gives $m'^{-1} m m' K = mK$. The next line (2) also requires $m \in M^-$, which guarantees $I^+ \cap m\widetilde{K} m^{-}1 = m I^+ m^{-}1$. Hence[2],

$$\mathcal{U}_m a_{m'} = m' \sum_{i \in I^+ / m I^+ m^{-1}} i \cdot a_m.$$

Note that the summation above is by definition $\mathcal{U}_m a_1$. This proves the following

**Lemma** V.1.3.1. *Let $m \in M^-$ and $m' \in M$. We have*

$$\mathcal{U}_m a_{m'} = m' \cdot \mathcal{U}_m a_1.$$

*In other words, the operator $\mathcal{U}_m$ is $M$-invariant.*

Now, we extend the action of $\mathcal{U}_m$ to the set

$$\mathcal{B}^\circ_{\text{ext}} := G \cdot (a_\circ, 0) = B \cdot (a_\circ, 0),$$

the latter equality holds thanks to the Iwasawa decomposition $G = BK = B\widetilde{K}$.

**Lemma** V.1.3.2. *For each $g \in G$, set $a_g := g \cdot (a_\circ, 0)$[3]. The intersection*

$$\mathcal{A}^\circ_{\text{ext}} \cap U^+ \cdot a_g$$

*consists of a single[4] vertex $\{a_{m_g}\}$ for some $m_g \in M$, unique modulo $M^1$.*

*Proof.* The intersection is nonempty by the decomposition $G = U^+ M K$. Suppose that there exist two $m, m' \in M$ such that $U^+ a_m = U^+ a_{m'}$. Let $u \in U^+$ such that $u \cdot a_m = a_{m'}$, i.e. $u \cdot a_m = a_{m'} = m' m^{-1} \cdot a_m$, hence $m m'^{-1} u \in P_{\{a_m\}}$, using the decomposition (see propositions II.3.4.1 and II.3.6.1)

$$P_{\{m\cdot a_\circ\}} = N_{\{m\cdot a_\circ\}} U_{\{m\cdot a_\circ\}} = N_{\{m\cdot a_\circ\}} U^-_{\{m\cdot a_\circ\}} U^+_{\{m\cdot a_\circ\}}.$$

Write $(m m'^{-1}) u = n u_- u_+$ for some $n \in N_{\{m\cdot a_\circ\}}$ and $u_\pm \in U^\pm_{\{m\cdot a_\circ\}}$. But $N \cap U^+ U^- = \{1\}$

[2]Recall that $m I^+ m^{-1} \subset I^+$ again by Lemma III.7.0.3.
[3]This is compatible with our previous notation $a_m = m \cdot (a_\circ, 0)$ for $m \in M$.
[4]Actually, the extended apartment $\mathcal{A}_{\text{ext}}$ is a fundamental domain for the action of $U^+$ on $\mathcal{B}_{\text{ext}}$.

[BT65, 5.15], hence

$$mm'^{-1} = n \in N_{\{m\cdot a_\circ\}} \text{ and } u = u_+ \in U^+_{\{m\cdot a_\circ\}},$$

and so in the reduced building $a_{m'} = ua_m = a_m$. This proves the lemma.

An alternative argument: For each $u \in U^+$, the restriction of the action of $u^{-1}$ to $u \cdot \mathcal{A}_{\text{ext}}$ is the unique isomorphism (uniqueness resembles the proof of Lemma V.1.2.1) mapping $u \cdot \mathcal{A}_{\text{ext}}$ to $\mathcal{A}_{\text{ext}}$ and fixing their intersection pointwise. In particular, it fixes $a_{m'} = u \cdot a_m \in u \cdot \mathcal{A}_{\text{ext}} \cap \mathcal{A}_{\text{ext}}$, thus $a_m = a_{m'}$. □

**Lemma** V.1.3.3. *Let $m \in M^-$, $g \in G$ and $u_g \in U^+$ verifying $a_g = u_g a_{m_g}$. Define*

$$\mathcal{U}_m a_g := u_g \cdot \mathcal{U}_m a_{m_g} \in \mathbb{Z}[\mathcal{B}^\circ_{\text{ext}}].$$

*This definition does not depend on the chosen $u_g \in U^+$.*

*Proof.* By Lemma V.1.3.1, we have

$$\mathcal{U}_m a_g = u_g m_g \cdot \mathcal{U}_m a_1$$

Let $u \in U^+$ be another element such that $a_g = um_g \cdot (a_\circ, 0)$. Therefore, by propositions II.3.4.1 and II.3.6.1

$$m_g^{-1} u_g^{-1} u m_g \in P^+_{\{a_\circ\}} \cap G^1 \cap U^+ = U^+_{\{a_\circ\}} = I^+.$$

But $\mathcal{U}_m(a_\circ, 0) = I \cdot a_m$, hence

$$\mathcal{U}_m a_g = u_g m_g \cdot \mathcal{U}_m(a_\circ, 0) = um_g \cdot \mathcal{U}_m(a_\circ, 0). \quad \square$$

**Corollary** V.1.3.1. *For every $m \in M^-$, we have*

$$\mathcal{U}_m \in \mathrm{End}_{\mathbb{Z}[B]} \mathbb{Z}[\mathcal{B}^\circ_{\text{ext}}].$$

*Proof.* This is a straightforward consequence of Lemmas V.1.3.1 and V.1.3.3. □

**Remark** V.1.3.1. *By definition, the map*

$$M^- \longrightarrow End_{\mathbb{Z}[B]} \mathbb{Z}[\mathcal{B}^\circ_{\text{ext}}], \quad m \longmapsto \mathcal{U}_m$$

*factorizes through the quotient $\underline{\Lambda}^-_M := M^-/M^1 \subset \Lambda_M/(\Lambda_M)_{tor}$, since we have $\mathcal{U}_m(a_\circ, 0) = \tilde{I}m \cdot (a_\circ, 0)$ and $M^1 \subset \tilde{I}$ (§III.15.1).*

**Lemma** V.1.3.4. *For every $m, m' \in \underline{\Lambda}^-_M$, we have*

$$\mathcal{U}_m \circ \mathcal{U}_{m'} = \mathcal{U}_{m+m'}.$$

*In particular, the operators $\mathcal{U}_m$ and $\mathcal{U}_{m'}$ commute.*

*Proof.* Here we will abuse notation and use the same letter for a class in $\underline{\Lambda}_M^-$ and a representative for it in $M$.

Recall that the Iwasawa decomposition $G = BK$ yields $\mathcal{B}_{\text{ext}}^{\circ} = B \cdot (a_\circ, 0)$. Thus, by the $B$-equivariance of the operators $\mathcal{U}_m$ and $\mathcal{U}_{m'}$, it suffices to verify $\mathcal{U}_m \circ \mathcal{U}_{m'} a_1 = \mathcal{U}_{m+m'} a_1$ for $m, m' \in \underline{\Lambda}_M^-$. We thus have

$$\begin{aligned}
\mathcal{U}_m \circ \mathcal{U}_{m'} a_1 &= \mathcal{U}_m \sum_{i' \in I^+/m'I^+m'^{-1}} i'm' \cdot (a_\circ, 0) \\
&= \sum_{i' \in I^+/m'I^+m'^{-1}} i'm'\mathcal{U}_m \cdot (a_\circ, 0) && (B\text{-equivariance}) \\
&= \sum_{i' \in I^+/m'I^+m'^{-1}} \sum_{i \in I^+/mI^+m^{-1}} i'm'im \cdot (a_\circ, 0)
\end{aligned}$$

Since $m, m'$ are chosen in $M^-$, we have

$$m'mI^+m^{-1}m'^{-1} \subset mI^+m^{-1} \subset I^+$$

so if $\mathcal{S}, \mathcal{S}' \subset I^+$ is a set of representatives for $I^+/mI^+m^{-1}$ resp. $I^+/m'I^+m'^{-1}$ , then

$$\mathcal{S}'' = \{i'm'im'^{-1} \colon i' \in \mathcal{S}' \text{ and } i \in \mathcal{S}\}$$

is a set of representatives for $I^+/m'mI^+(m'm)^{-1}$. Therefore,

$$\mathcal{U}_m \circ \mathcal{U}_{m'} a_\circ = \sum_{i'' \in \mathcal{S}''} i'' \cdot a_{m+m'} = \mathcal{U}_{m+m'}(a_\circ, 0). \quad \square$$

**Remark** V.1.3.2. *Note that the definition of the above operators $\{\mathcal{U}_m \colon m \in \underline{\Lambda}_M^-\}$ is independent from the alcove $\mathfrak{a}$ and the special point $a_\circ$. Actually, such operators should be imagined (should be defined in the first place) as "successor" operators with respect to a point lying in the building at infinity.*

**Definition** V.1.3.1. *Define $\mathcal{U} \subset \mathrm{End}_{\mathbb{Z}[B]}\mathbb{Z}[\mathcal{B}_{\text{ext}}^{\circ}]$ to be the ring generated by the operators $\{\mathcal{U}_m \colon m \in \underline{\Lambda}_M^-\}$.*

**Corollary** V.1.3.2. *The following map is a natural isomorphism of rings*

$$\mathbb{Z}[\underline{\Lambda}_M^-] \xrightarrow{\simeq} \mathcal{U}, \qquad m \longmapsto \mathcal{U}_m.$$

*Proof.* We claim that the kernel of the map $M^- \longrightarrow \mathcal{U}$ given by $m \mapsto \mathcal{U}_m$ is precisely $M^1$. Indeed, we have:

1. The only points of the apartment where the alcove based retraction $r_{\mathcal{A}_{\text{ext}},\mathfrak{a}}$ has trivial fiber (i.e. reduced to the point itself) is the closure of the alcove $\mathfrak{a}$, and this true by Lemma V.1.2.2.

2. The closure of the alcove $\mathfrak{a} \times V_G$ contains a single point of $M^- \cdot (a_\circ, 0)$, namely $(a_0, 0)$ itself. Indeed, as in Example II.3.4.1 we know that
$$\bar{\mathfrak{a}} = \{a \in \mathbb{A}_{\mathrm{red}}(\mathbf{G}, \mathbf{S}) \colon 0 \leq \alpha(a - a_\circ) \leq n_\alpha^{-1} \text{ for all } \alpha \in \Phi^+_{\mathrm{red}}\}.$$
But $M^- = \{m \in M \colon \alpha(\nu(m) + a_\circ) \leq 0,\ \forall \alpha \in \Phi^+_{\mathrm{red}}\}$, thus
$$\bar{\mathfrak{a}} \cap (M^- \cdot (a_\circ, 0)) = \{(a_\circ, 0)\}.$$

Combining the above claim with Lemma V.1.3.4 shows the corollary. □

**Lemma** V.1.3.5. *The geometric action of $\mathcal{U}$ on $\mathbb{Z}[\mathcal{B}^\circ_{\mathrm{ext}}]$ is faithful.*

*Proof.* This is morally similar to the proof of Lemma III.14.2.2. Let $m_1, \cdots, m_r \in \underline{\Lambda}_M^-$ distinct and $s_1, \cdots, s_r \in \mathbb{Z}$. We then have
$$\begin{aligned} \widetilde{K} \cdot \sum_i s_i \mathcal{U}_{m_i} a_1 &= \sum_i s_i\, \widetilde{K} \cdot I \cdot a_{m_i} \\ &= \sum_i s_i\, (\widetilde{K} m_i \widetilde{K}) \cdot a_1 \end{aligned}$$
Hence, if $\widetilde{K} \cdot \sum_i s_i \mathcal{U}_m a_1 = a_1$, by Cartan decomposition (Proposition III.4.0.1), one must have
$$s_i = \begin{cases} 1 & \text{If } m_i = 1 \in \Lambda_M^-, \\ 0 & \text{If } m_i \neq 1 \in \Lambda_M^-. \end{cases} \quad \square$$
Combining Lemmas V.1.3.1, V.1.3.4 and V.1.3.5 we get a natural embedding of rings
$$\mathcal{U} \hookrightarrow \mathrm{End}_{\mathbb{Z}[B]} \mathbb{Z}[\mathcal{B}^\circ_{\mathrm{ext}}]. \tag{V.1}$$

**Theorem** V.1.3.1. *The subring $\mathcal{U} \subset \mathrm{End}_{\mathbb{Z}[B]} \mathbb{Z}[\mathcal{B}^\circ_{\mathrm{ext}}]$ is integral over $\mathrm{End}_{\mathbb{Z}[G]} \mathbb{Z}[\mathcal{B}^\circ_{\mathrm{ext}}]$.*

*Proof.* We have by definition of $\widetilde{K}$
$$G/\widetilde{K} \simeq \mathcal{B}^\circ_{\mathrm{ext}} \simeq \mathcal{B}^\circ_{\mathrm{red}} \times \mathbb{Z}^{d_{Z,sp}}$$
where $d_{Z,sp} := \mathrm{rk}(\mathbf{Z}_{c,sp})$ is the split rank of the maximal split central torus (§II.3.7). The homomorphism of rings $\widetilde{\mathbb{U}} = \dot{\widetilde{\Theta}}_{\mathrm{Bern}}(\mathbb{Z}[\underline{\Lambda}_M^-]) \simeq \mathcal{U}$ given on basis elements by $\mathcal{U}_m \mapsto \widetilde{\imath}_m = \mathbf{1}_{\widetilde{I}m\widetilde{I}}$ for all $m \in \underline{\Lambda}_M^-$ is an isomorphism (See III.15.5). This identification between $\widetilde{\mathbb{U}}$ and $\mathcal{U}$ is compatible with their corresponding actions on $\mathcal{C}_c(G/\widetilde{K}, R)$ and $R[\mathcal{B}^\circ_{\mathrm{ext}}]$. Indeed, on the one hand we have
$$\mathcal{U}_m a_\circ = \sum_{i \in \widetilde{I}^+/m\widetilde{I}^+m^{-1}} i \cdot a_m$$
and on the other hand we have by definition of the excursion pairing of §III.15.5:
$$\mathbf{1}_{\widetilde{K}} \bullet \widetilde{\imath}_m = \sum_{i \in \widetilde{I}^+/m\widetilde{I}^+m^{-1}} \mathbf{1}_{i\widetilde{K}}.$$

The following diagram is commutative:

$$\begin{tikzcd}
\widetilde{\mathbb{U}}^{\mathrm{opp}} \arrow[r, "\simeq", "\dot{\Theta}_{\mathrm{Bern}}^{-1}"'] \arrow[d, hook, "\text{Thrm. III.15.5.1}"'] & \mathbb{Z}[\underline{\Lambda}_M^-] \arrow[r, "\simeq", "\text{Cor. V.1.3.2}"'] & \mathcal{U} \arrow[d, hook, "(V.1)"] \\
\mathrm{End}_B\,\mathcal{C}_c(G/\widetilde{K},\mathbb{Z}) \arrow[rr, "\simeq"] & & \mathrm{End}_{\mathbb{Z}[B]}\,\mathbb{Z}[\mathcal{B}^\circ_{\mathrm{ext}}]
\end{tikzcd}$$

The lower horizontal map is defined as follows: By Iwasawa decomposition $G = B\widetilde{K}$ an element $u$ in $\mathrm{End}_B\,\mathcal{C}_c(G/\widetilde{K},\mathbb{Z})$ is defined by the value it gives to the function $\mathbf{1}_{\widetilde{K}}$, say $\sum_i a_i \mathbf{1}_{g_i\widetilde{K}}$. The endomorphism $u$ is then sent to the unique endomorphism[5] $u\prime \in \mathrm{End}_{\mathbb{Z}[B]}\,\mathbb{Z}[\mathcal{B}^\circ_{\mathrm{ext}}]$ sending $a_\circ$ to $\sum_i a_i g_i \cdot a_\circ$.

The $G$-equivariant bijection $G/\widetilde{K} \to \mathcal{B}^\circ_{\mathrm{ext}}$ yields a $G$-equivariant isomorphism $\mathcal{C}_c(G/\widetilde{K},\mathbb{Z}) \to \mathbb{Z}[\mathcal{B}^\circ_{\mathrm{ext}}]$, an isomorphism of rings $\mathrm{End}_B(\mathcal{C}_c(G/\widetilde{K},\mathbb{Z})) \to \mathrm{End}_B(\mathbb{Z}[\mathcal{B}^\circ_{\mathrm{ext}}])$, which identifies the subrings $\mathcal{U}$ and $\mathbb{U}$, $\mathrm{End}_G(\mathcal{C}_c(G/\widetilde{K},\mathbb{Z}))$ and $\mathrm{End}_G(\mathbb{Z}[\mathcal{B}^\circ_{\mathrm{ext}}])$. The covariance (in $\mathcal{B}^\circ_{\mathrm{ext}} \to \mathbb{Z}[\mathcal{B}^\circ_{\mathrm{ext}}]$) and contravariance (in $G/K \to \mathcal{C}_c(G/K)$) explains the use of the opposite ring in the upper horizontal arrow. Accordingly, the integrality of $\widetilde{\mathbb{U}}$ over $\mathrm{End}_G\,\mathcal{C}_c(G/\widetilde{K},\mathbb{Z})$ (Corollary III.15.5.1) is equivalent to the integrality of $\mathcal{U}$ over $\mathrm{End}_{\mathbb{Z}[G]}\,\mathbb{Z}[\mathcal{B}^\circ_{\mathrm{ext}}]$. $\square$

## V.2 Filtrations and $\mathcal{U}$-operators

In this section we will present yet another alternative point of view for the geometric operators ring $\mathcal{U}$, this was suggested by C. Cornut. For notation we refer the reader to chapter V.

### V.2.1 Definition

Set $\mathbf{F} = \mathbf{F}(G)$ to be the set of all $\mathbb{R}$-filtrations on $G = \mathbf{G}(F)$ [Cor17, page 77], this is the vectorial Tits building $\mathbf{F}$ [Cor17, Chapter 4]. By [Cor17, §6.2], the extended Bruhat–Tits building $\mathcal{B}_{\mathrm{ext}} = \mathcal{B}(\mathbf{G}, F)_{\mathrm{ext}}$ is an affine $\mathbf{F}$-space [Cor17, §5.2.1.], in particular there is a $G$-equivariant right "action" of the vectorial Tits building $\mathbf{F}$ on $\mathcal{B}_{\mathrm{ext}}$,

$$\mathcal{B}_{\mathrm{ext}} \quad \times \quad \mathbf{F} \longrightarrow \mathcal{B}_{\mathrm{ext}} \qquad (a,\mathcal{F}) \longmapsto a + \mathcal{F}.$$

[5]Such an isomorphism exists because the required image is fixed by $B \cap \widetilde{K}$.

Define $\alpha_{\mathcal{F}}(a) = a + \mathcal{F}$, and let $\mathbf{F}^\circ$ be the subset of all filtrations $\mathcal{F} \in \mathbf{F}$ such that $\alpha_{\mathcal{F}}(\mathcal{B}^\circ_{\text{ext}}) = \mathcal{B}^\circ_{\text{ext}}$. Thus we get a right "action":

$$\mathcal{B}^\circ_{\text{ext}} \quad \times \quad \mathbf{F}^\circ \longrightarrow \mathcal{B}^\circ_{\text{ext}} \qquad (a, \mathcal{F}) \longmapsto \alpha_{\mathcal{F}}(a) = a + \mathcal{F}.$$

**Example** V.2.1.1. [Cor17, §6.1] *Let $\mathcal{V} \neq 0$ be a $F$-vector space of dimension $n \in \mathbb{N}$, and $\mathbf{G} = \mathbf{GL}(\mathcal{V})$. Set*

$$\mathbf{S}(\mathcal{V}) := \{\mathcal{S} \subset \mathbb{P}^1(\mathcal{V})(F) \colon \mathcal{V} = \oplus_{L\in\mathcal{S}} L\},$$
$$\mathbf{F}(\mathcal{V}) := \{\mathbb{R}\text{-filtrations on } V\}.$$

*A $F$-norm on $\mathcal{V}$ is said to be splittable by $\mathcal{S} \in \mathbf{S}(\mathcal{V})$ if and only if*

$$\forall v \in \mathcal{V}\colon \quad \alpha(v) = \max\{\alpha(v_L) \colon L \in \mathcal{S}\}, \ \text{where } v = \sum_{L\in\mathcal{S}} v_L, v_L \in L.$$

*The extended building $\mathcal{B}(\mathbf{GL}(\mathcal{V}), F)_{\text{ext}}$ identifies naturally with $\mathcal{B}(\mathcal{V})$; the set of all splittable (by some $\mathcal{S} \in \mathbf{S}(\mathcal{V})$) $F$-norms on $\mathcal{V}$. The group $G$ acts on $\mathcal{B}(\mathcal{V})$ by $(g \cdot \alpha)(v) = \alpha(g^{-1}v)$, for every $g \in G$ and every splittable $F$-norm $\alpha$ on $\mathcal{V}$. The action of $\mathbf{F}(G) \simeq \mathbf{F}(\mathcal{V})$ on $\mathcal{B}(\mathcal{V})$ is described as follows: For any $\mathcal{F} \in \mathbf{F}(\mathcal{V})$ and $\alpha \in \mathcal{B}(\mathcal{V})$, there exists $\mathcal{S} \in \mathbf{S}(\mathcal{V})$ such that $\alpha$ is splittable by $\mathcal{S}$ and $\mathcal{F} \in \mathbf{F}(\mathcal{S})$, the action is then given by*

$$(\alpha + \mathcal{F})(v) = \max\left\{e^{-\mathcal{F}(L)}\alpha(v_L) \colon L \in \mathcal{S}\right\}.$$

**Definition** V.2.1.1. *Define*

$$\beta_{\mathcal{F}}(b) = \sum_{\alpha_{\mathcal{F}}(a)=b} a.$$

Let $P_{\mathcal{F}}$ be the stabilizer of $\mathcal{F}$ in $G$, this is the group of $F$-points of a parabolic subgroup of $\mathbf{G}$ [Cor17, §2.2.8]. Then by $G$-equivariance we see that

$$\alpha_{\mathcal{F}}, \beta_{\mathcal{F}} \in \operatorname{End}_{P_{\mathcal{F}}}\left(\mathbb{Z}[\mathcal{B}^\circ_{\text{ext}}]\right),$$

Set $\mathbf{F}^\circ(B) = \{\mathcal{F} \in \mathbf{F}^\circ \colon B \subset P_{\mathcal{F}}\}$, then

**Lemma** V.2.1.1. *For any $\mathcal{F}, \mathcal{G}$ in $\mathbf{F}^\circ(B)$, we have*

$$\alpha_{\mathcal{F}} \circ \alpha_{\mathcal{G}} = \alpha_{\mathcal{G}} \circ \alpha_{\mathcal{F}} \ \text{and} \ \beta_{\mathcal{F}} \circ \beta_{\mathcal{G}} = \beta_{\mathcal{G}} \circ \beta_{\mathcal{F}} \ \text{in} \ \operatorname{End}_B(\mathbb{Z}[\mathcal{B}^\circ_{\text{ext}}]).$$

*Proof.* The first commutativity rule results from the condition AC [Cor17, §5.2.5], this condition ensures that for any $a \in \mathcal{B}^\circ_{\text{ext}}$

$$\alpha_{\mathcal{G}} \circ \alpha_{\mathcal{F}}(a) = (a + \mathcal{F}) + \mathcal{G} = a + (\mathcal{F} + \mathcal{G}) = (a + \mathcal{G}) + \mathcal{F} = \alpha_{\mathcal{F}} \circ \alpha_{\mathcal{G}}(a).$$

The same condition *loc. cit* also yields

$$\begin{aligned}\beta_{\mathcal{F}} \circ \beta_{\mathcal{G}}(b) &= \beta_{\mathcal{F}}\left(\sum_{\alpha_{\mathcal{G}}(a)=b} a\right)\\ &= \sum_{\alpha_{\mathcal{G}}(a)=b}\ \sum_{\alpha_{\mathcal{F}}(c)=a} c\\ &= \sum_{\alpha_{\mathcal{G}+\mathcal{F}}(c)=b} c\\ &= \sum_{\alpha_{\mathcal{F}+\mathcal{G}}(c)=b} c = \beta_{\mathcal{G}} \circ \beta_{\mathcal{F}}(b). \quad \square\end{aligned}$$

### V.2.2 Formula

**Lemma** V.2.2.1. *Let $\mathcal{F} \in \mathbf{F}^\circ$, $b \in \mathcal{B}^\circ_{\text{ext}}$ and fix $a \in \mathcal{B}^\circ_{\text{ext}}$ such that $\alpha_{\mathcal{F}}(a) = b$, then*

$$\beta_{\mathcal{F}}(b) = \sum_h ha, \quad h \in U_{\mathcal{F}} \cap \mathrm{Stab}(b)/U_{\mathcal{F}} \cap \mathrm{Stab}(a).$$

*Proof.* For $x \in \mathcal{B}_{\text{ext}}$ and $\mathcal{F} \in \mathbf{F}^\circ$, we may consider the geodesic ray [Cor17, §5.2.11]

$$\mathbb{R}_+ \longrightarrow \mathcal{B}_{\text{ext}}, \quad t \longmapsto x + t\mathcal{F}.$$

Since we are working over a complete field, any geodesic ray is standard meaning it is contained in some apartment [Cor17, 6.2.8]. Thus, for any $a' \in \alpha_{\mathcal{F}}^{-1}(b)$, there are apartments $\mathcal{A}$ and $\mathcal{A}'$ containing respectively $\{a + t\mathcal{F}, t \geq 0\}$ and $\{a' + t\mathcal{F}, t \geq 0\}$.

Suppose that $a + \mathcal{F} = a' + \mathcal{F} = b \in \mathcal{A} \cap \mathcal{A}'$ ad enxtend the half lines $a + t\mathcal{F}$ and $a' + t\mathcal{F}$ to geodesic lines $L$ and $L'$, given by $b + t\mathcal{G}$ and $b + t\mathcal{G}'$, for $\mathcal{G}$ and $\mathcal{G}'$ opposed to $\mathcal{F}$. There exists a $u$ in $U_{\mathcal{F}}$ (the unipotent radical of $P_{\mathcal{F}}$) mapping $\mathcal{G}$ to $\mathcal{G}'$. It also fixes $a + t\mathcal{F} = b + (t-1)\mathcal{F} = a' + t\mathcal{F}$ for $t \gg 0$. It follows that $u$ maps $L$ to $L'$ and fixes their intersection[6], which contains $b + t\mathcal{F}$ for all $t \geq 0$. It then maps $a = b + \mathcal{G}$ to $a' = b + \mathcal{G}'$. This proves the lemma. $\square$

[6]One may be surprised by the fact that any $u$ mapping $\mathcal{G}$ to $\mathcal{G}'$ fixes $b$. This is because $\mathcal{G}$ and $\mathcal{G}'$ are not merely "opposed to $\mathcal{F}$" in the building at infinity, they are actually, and by construction, "opposed to $\mathcal{F}$ at $b$", i.e. $b + t\mathcal{G}$ and $b + t\mathcal{F}$ ($t \geq 0$) make a $\pi$-angle at $b$.

### V.2.3 Comparison with $\mathcal{U}$

Recall that the extended standard apartment $\mathbb{A}(\mathbf{G},\mathbf{S})_{\mathrm{ext}}$ is an affine space over the real vector space $X_*(\mathbf{S})\otimes_{\mathbb{Z}}\mathbb{R}$, and $m\in M$ acts on it by translation $\nu_M(m)$ (Lemma II.3.2.2). Likewise, the vector space $X_*(\mathbf{S})\otimes_{\mathbb{Z}}\mathbb{R}$ is canonically isomorphic to the apartment $\mathbf{F}(S)$ of $\mathbf{F}$ corresponding to $\mathbf{S}$ [Cor17, Definition 9 and §4.1.13 ]. Using this identification, for each $m\in M^-$ set $\mathcal{F}_m\in\mathbf{F}(S)\subset\mathbf{F}$ for the filtration corresponding [7] to $-\nu_M(m)\in X_*(\mathbf{S})\otimes_{\mathbb{Z}}\mathbb{R}$. Then

$$\alpha_{\mathcal{F}_m},\beta_{\mathcal{F}_m}\in\mathrm{End}_B(\mathbb{Z}[\mathcal{B}^\circ_{\mathrm{ext}}]).$$

**Theorem** V.2.3.1. *Let $m\in M^-$, then we have*

$$\beta_{\mathcal{F}_m}=\mathcal{U}_m\ \textit{in}\ \mathrm{End}_B(\mathbb{Z}[\mathcal{B}^\circ_{\mathrm{ext}}]).$$

*Proof.* Let $\widetilde{K}$ be the stabilizer of the distinguished base point $(a_\circ,0)\in\mathcal{B}^\circ$. This is the open compact subgroup (§II.3.7)

$$\widetilde{K}=P_{a_\circ}\cap\ker\nu_G=P_{a_\circ}\cap G^1.$$

By §II.3.9.2, it is also the group of elements $g\in P_{a_\circ}$ such that $\kappa_G(g)$ is torsion, hence $K$ has finite index in $\widetilde{K}$.

By the formula of Lemma V.2.2.1, we have

$$\beta_{\mathcal{F}_m}(a_1)=\sum_h hma_1,\quad h\in U_{\mathcal{F}_m}\cap\widetilde{K}/U_{\mathcal{F}_m}\cap m\widetilde{K}m^{-1}.$$

**Lemma** V.2.3.1. *For any closed subset*[8] *of roots in $\Psi\subset\Phi_+\cap\Phi_{\mathrm{red}}$ let $\mathbf{U}_\Psi$ be the unique closed, connected, unipotent subgroup of $\mathbf{G}$ given by Proposition II.2.5.1. We have*

$$U_\Psi\cap\widetilde{K}=U_\Psi\cap K=\prod_{\alpha\in\Psi}^{\prec}U_{\alpha+0},$$

*for any fixed ordering $\prec$ on $\Psi$.*

*proof of Lemma V.2.3.1.* By the proof of [HR10, Lemma 4.1.1] we have $\widetilde{K}=(M\cap\widetilde{K})K$. Let $\Psi$ be any closed subset of roots in $\Phi_+\cap\Phi_{\mathrm{red}}$. Let $u\in U_\Psi\cap\widetilde{K}$, and write it as $m_u u_- u_+$,

---

[7] We want $m\cdot a_1+\mathcal{F}_m=a_1$, i.e. $a_1+\nu(m)+\mathcal{F}_m=a_1=(a_\circ,0)$ in the apartment $\mathcal{A}_{\mathrm{ext}}=\mathcal{A}_{\mathrm{ext}}(\mathbf{S})$, so we want $\nu(m)+\mathcal{F}_m=0$, i.e. $\mathcal{F}_m$ is opposed to $\nu(m)$. Alternatively: $\mathcal{F}_m$ corresponds to $m^{-1}$, which is in $M^+$.

[8] Recall that $\Psi$ is closed if for any $\alpha,\beta\in\Psi$ one has

$$[\alpha,\beta]:=\{n\alpha+m\beta\colon\ \text{for all}\ n,m\in\mathbb{Z}_{>0}\}\cap\Phi\subset\Psi.$$

with $u_+ \in (U^+ \cap K)$, $u_- \in (U^- \cap K)$. Thus, $(uu_+^{-1})(u_-)^{-1} = m_u \in U^+U^- \cap N = \{1\}$ [BT65, 5.15], we must have

$$u = u_+ \in U_\Psi \cap \widetilde{K} \cap U^+ \cap K = (U_\Psi \cap U^+) \cap (\widetilde{K} \cap K) = U_\Psi \cap K.$$

Hence, $U_\Psi \cap \widetilde{K} = U_\Psi \cap K$.

By proposition II.3.4.1 (2) and Example II.3.4.1, we have $U_\alpha \cap K = U_{\alpha+0}$ for any $\alpha \in \Phi_{\text{red}}$. Combining this with (c) Proposition II.2.5.1 and with Corollary II.3.9.2 we obtain the desired

$$U_\Psi \cap \widetilde{K} = \prod_{\alpha\in\Psi}^{\prec} U_{\alpha+0}$$

for any fixed ordering $\prec$ on $\Psi$. This shows the lemma. □

**Remark** V.2.3.1. *This proof gives actually a slightly more general result. For any bounded subset $\Omega \subset \mathcal{B}_{\text{red}}$ (e.g. facets). Let $K_\Omega$ be the parahoric associated to $\Omega$ i.e. its pointwise stabilizer in $G_1$, and $\widetilde{K}_\Omega$ its pointwise stabilizer in $G^1$. We thus have*

$$U_\Psi \cap \widetilde{K}_\Omega = U_\Psi \cap K_\Omega,$$

*for any closed subset $\Psi \subset \Phi_+ \cap \Phi_{\text{red}}$.*

Let $m \in M^-$, then $P_{\mathcal{F}_m} = U_{\mathcal{F}_m} \rtimes M_{\mathcal{F}_m}$ is a *semi-standard* (§II.2.6) parabolic subgroup containing the minimal parabolic subgroup $B$. Let $\Psi_m$ be the closed subset of roots in $\Phi_+ \cap \Phi_{\text{red}}$ corresponding to $U_{\mathcal{F}_m} = U_{\Psi_m}$, then by the above lemma

$$U_{\mathcal{F}_m} \cap K = \prod_{\alpha\in\Psi_m}^{\prec} U_{\alpha+0}$$

for any fixed ordering $\prec$ of $\Psi_m$.

**Lemma** V.2.3.2. *The inclusion $U_{\mathcal{F}_m} \cap K \hookrightarrow I^+ = U^+ \cap K$, induces the bijection*

$$U_{\mathcal{F}_m} \cap K / U_{\mathcal{F}_m} \cap mKm^{-1} \xrightarrow{\simeq} I^+/mI^+m^{-1}.$$

*proof of Lemma V.2.3.2.* We have $\mathbf{U}^+ = \mathbf{U}_{\mathcal{F}_m}(\mathbf{U}^+ \cap \mathbf{M}_{\mathcal{F}_m})$, so by Lemma V.2.3.1 we get a decomposition

$$U^+ \cap K = (U^+ \cap M_{\mathcal{F}_m} \cap K)(U_{\mathcal{F}_m} \cap K) = (U_{\mathcal{F}_m} \cap K)(U^+ \cap M_{\mathcal{F}_m} \cap K). \tag{V.2}$$

By Lemma II.3.4.1, we have for all $\alpha \in \Phi_{\text{red}}$

$$mU_{\alpha+0}m^{-1} = U_{\alpha-\langle\alpha,\nu(m)\rangle},$$

but since $\langle\alpha,\nu(m)\rangle = 0$ for all $\alpha \in \Phi(\mathbf{M}_{\mathcal{F}_m}, \mathbf{S})$[9] we have

$$U^+ \cap M_{\mathcal{F}_m} \cap mKm^{-1} = U^+ \cap M_{\mathcal{F}_m} \cap K. \tag{V.3}$$

[9]By definition, the semi-standard Levi $\mathbf{M}_{\mathcal{F}}$ centralizes $m$.

Now consider the natural projection map ($U_\Psi \subset U^+$)

$$U_{\mathcal{F}_m} \cap K \to U^+ \cap K / U^+ \cap mKm^{-1}.$$

We claim that it is surjective. Let $u \in U^+ \cap K$ be written as $u = u_1 u_2$ with $u_1 \in U_{\mathcal{F}_m} \cap K$ and $u_2 \in U^+ \cap M_{\mathcal{F}_m} \cap K$, therefore

$$\begin{aligned} u_1u_2(U^+ \cap mKm^{-1}) &= u_1u_2(U^+ \cap M_{\mathcal{F}_m} \cap mKm^{-1})(U_{\mathcal{F}_m} \cap mKm^{-1}) && \text{by (V.2)} \\ &= u_1u_2(U^+ \cap M_{\mathcal{F}_m} \cap K)(U_{\mathcal{F}_m} \cap mKm^{-1}) && \text{by (V.3)} \\ &= u_1(U^+ \cap M_{\mathcal{F}_m} \cap K)(U_{\mathcal{F}_m} \cap mKm^{-1}) \\ &= u_1(U^+ \cap mKm^{-1}), \end{aligned}$$

which proves the claim. Finally it is clear that the kernel of the above map is precisely $U_{\mathcal{F}_m} \cap mKm^{-1}$. Which proves the lemma, since by the Iwahori factorization we have $I^+ = U^+ \cap K$, and we also have $mI^+m^{-1} = U^+ \cap mKm^{-1}$ for $m \in M^-$. □

In conclusion, using the above lemmas, we can rewrite the formula of lemma V.2.2.1 as follows $\beta_{\mathcal{F}_m}(a_\circ) = \sum_{h \in I^+/mI^+m^{-1}} hma_\circ$, and this shows by $B$-equivariance:

$$\beta_{\mathcal{F}_m} = \mathcal{U}_m. \quad \square$$

# V.3 Norm-compatible families of vertices

We construct in this section some families of metric trees embedded in the extended building of $\mathbf{G}$. We will use these metric graphs to obtain what we call norm compatible systems of vertices.

## V.3.1 Families of trees

Set $a_1 = (a_\circ, 0)$, and for every $m \in M$ we will continue to use the notation $a_m = m{\cdot}(a_\circ, 0) \in \mathcal{A}^\circ_{\text{ext}}$.

**Definition** V.3.1.1. *Let $M^{--}$ be the subset of antidominant elements $m \in M^-$ verifying the following two conditions*

1. *The intersection of the geodesic segment*[10]

$$[a_1, a_m] \cap \mathcal{A}^\circ_{\mathrm{ext}}$$

*consists only of the two points $a_1$ and $a_m$.*

2. *For all $\alpha \in \Phi^+_{\mathrm{red}}$*

$$(\alpha + 0)(a_\circ + \nu_M(m)) = \langle \nu_M(m), \alpha \rangle < 0.$$

**Lemma** V.3.1.1. *Fix $m \in M^{--}$. For every $u \in U^+ \backslash \{1\}$, there exists a integer $t_u$ verifying*

$$\{t \in \mathbb{Z} \colon u \cdot a_{m^t} = a_{m^t}\} = \{t \in \mathbb{Z} \colon t \leq t_u\}.$$

**Remark** V.3.1.1. *Geometrically: The $a_{m^t}$'s are on a line and the fixed point set of $u$ is convex, so all we need to show is that (1) $u$ fixes $a_{m^t}$ for $t \ll 0$ and (2) $u$ does not fix $a_{m^t}$ for $t \gg 0$. For (1), this works for all $m \in M^-$. Then (2) follows from (1) and the regularity assumption on $m$, which says that the line is regular (= not contained in any affine root).*

*Proof.* When the group is semi-simple simply connected a slightly different statement can be found in [Leu03, Lemma 1]. Here, we prove it for any reductive group. Recall that we have

$$U^+ = \prod_{\alpha \in \Phi_{\mathrm{red}} \cap \Phi^+} U_\alpha,$$

for any fixed ordering $\Phi_{\mathrm{red}} \cap \Phi^+$. Write $u = \prod_{\alpha \in \Phi_{\mathrm{red}} \cap \Phi^+} u_\alpha$. The lemma claims that there exists $t_u$ such that: for any $t \in \mathbb{Z}$ we have

$$m^{-t} u m^t = \prod_{\alpha \in \Phi_{\mathrm{red}} \cap \Phi^+} m^{-t} u_\alpha m^t \in \widetilde{K} \quad \text{if and only if} \quad t \leq t_u.$$

By the Iwahori factorization of $\widetilde{K}$, this is equivalent to

$$m^{-t} u_\alpha m^t \in \widetilde{K}, \forall \alpha \in \Phi_{\mathrm{red}} \cap \Phi^+ \quad \text{if and only if} \quad t \leq t_u.$$

Now, by definition of the affine root groups (§II.3.3), we have for every $\alpha \in \Phi_{\mathrm{red}} \cap \Phi^+$ a filtration

$$U_\alpha = \cup_{r \in \Gamma_\alpha} U_{\alpha + r},$$

where $\Gamma_\alpha = n_\alpha^{-1}\mathbb{Z}$ for some integer $n_\alpha$ by Proposition II.3.3.2. There exists then a unique $t_\alpha \in \mathbb{Z}$ such that

$$u_\alpha \in U_{\alpha + t_\alpha n_\alpha^{-1}} \setminus U_{\alpha + (t_\alpha + 1) n_\alpha^{-1}}.$$

Therefore, for any $t'_\alpha \in \mathbb{Z}$, we have (Lemma II.3.4.1)

$$m^{-t'_\alpha} u_\alpha m^{t'_\alpha} \in m^{-t'_\alpha} U_{\alpha + t_\alpha n_\alpha^{-1}} m^{t'_\alpha} = U_{\alpha + t_\alpha n_\alpha^{-1} + t'_\alpha \langle \nu_M(m), \alpha \rangle}$$

[10] For rank one groups, that leaves just a minuscule cocharacter.

Thus, we have

$$t'_\alpha \leq t_{u_\alpha} := \left[\frac{t_\alpha n_\alpha^{-1}}{\langle \nu_M(m), \alpha\rangle}\right],$$

if and only if $m^{-t'_\alpha} u_\alpha m^{t'_\alpha} \in U_{\alpha+0} \subset \widetilde{K}$, i.e. $u_\alpha \in \mathrm{Stab}_G(a_{m^{t'_\alpha}}) = P_{m^{t'_\alpha}\cdot a_\circ} \cap G^1$. In conclusion, the above discussion shows that the integer

$$t_u = \min\{t_{u_\alpha} \colon \forall \alpha \in \Phi_{\mathrm{red}} \cap \Phi^+\} \in \mathbb{Z},$$

verifies the claim of the Lemma. □

Condition (2) in Definition V.3.1.1 implies that for any $m \in M^{--}$ the lattice $m^{\mathbb{Z}} \cdot a_1 := \{a_{m^t} \colon t \in \mathbb{Z}\}$ is contained in a unique geodesic line that we denote by $c_m \colon \mathbb{R} \to \mathcal{A}_{\mathrm{ext}}$. We normalize the map $c_m$ such that $c_m(\mathbb{Z}) = m^{\mathbb{Z}} \cdot a_1$, this leaves two orientation and we choose the one given by $c_m(n) = a_m^n$. The action of $G$ being isometric ensures that for any $g \in G$, we obtain a new geodesic line $g \cdot c_m(\mathbb{R}) \subset \mathcal{B}_{\mathrm{ext}}$.

**Lemma** V.3.1.2. *Fix $m \in M^{--}$. For any pair $u, u' \in U^+$ with $u \neq u'$, there is an integer $t_{u,u'}$ such that*

$$u \cdot c_m(\mathbb{Z}) \cap u' \cdot c_m(\mathbb{Z}) = \{u \cdot a_{m^t} \colon t \leq t_{u,u'}\}.$$

*In particular, the two geodesics $u \cdot c_m(\mathbb{Z})$ and $u' \cdot c_m(\mathbb{Z})$ do not coincide globally as long as $u \neq u'$.*

*Proof.* When the group $\mathbf{G}$ is semi-simple simply connected a slightly different statement can be found in [Leu03, Lemmas 2 & 3]. Here, we prove it for any reductive group. Firstly, notice that

$$u \cdot c_m(\mathbb{Z}) \cap u' \cdot c_m(\mathbb{Z}) = u\left(c_m(\mathbb{Z}) \cap u^{-1}u' \cdot c_m(\mathbb{Z})\right).$$

On the one hand, if there exists two integers $t'$ and $t''$ such that

$$a_{m^{t'}} = u^{-1}u' a_{m^{t''}} \in c_m(\mathbb{Z}) \cap u^{-1}u' \cdot c_m(\mathbb{Z}),$$

then by Lemma V.1.3.2 we must have that $t' = t''$. On the other hand, to conclude, we apply Lemma V.3.1.1

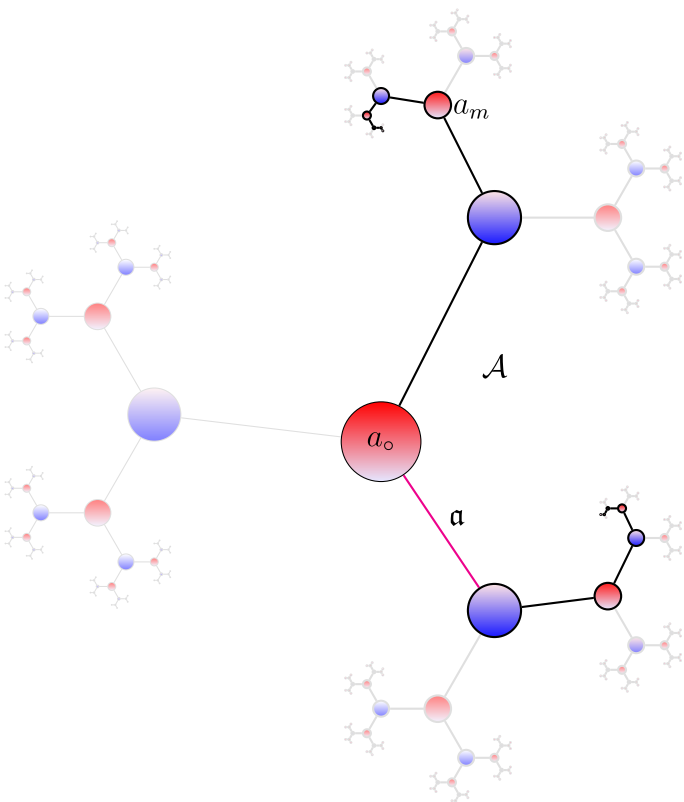
$$c_m(\mathbb{Z}) \cap u^{-1}u' \cdot c_m(\mathbb{Z}) = \{a_{m^t} \colon t \leq t_{u^{-1}u'}\}.$$

By the above discussion, we see that if $u \cdot c_m(\mathbb{Z}) = u' \cdot c_m(\mathbb{Z})$ then we can not have $u \neq u'$ otherwise the integer $t_{u,u'}$ can be chosen randomly big, thus we must have $u' = u$. This concludes the proof of the Lemma. □

Following Leuzinger [Leu03, §3.1], to each non-zero $m \in \underline{\Lambda}_M^-$ we associate a metric tree $\mathcal{T}_m \subset \mathcal{B}_{\mathrm{ext}}$. As a set the tree $\mathcal{T}_m$ is equal to $\{u \cdot c_m(\mathbb{R}) \colon u \in U^+\}$, its vertices are the points where the geodesics $u \cdot c_m(\mathbb{R})$, $u \in U^+$, "branch" and its edges are the segments between successive branching points. Geometrically: the tree is the union of all geodesic lines in the building extending some half line truncation $c_m(]\infty, t_0[)$ for some $t_0$. If $G$ is semi-simple of rank one, there is one choice for $m$ and $\mathcal{T}_m$ is the whole building (tree) .

**Remark** V.3.1.2. *Using the above Lemmas V.3.1.1 and V.3.1.2, one can drop the assumption* semisimple and simply connected *from Leuzinger's construction in his main theorem [Leu03, Theorem 1]. This will show that for every $m \in M^{--}$, the metric graph $\mathcal{T}_m$ is a locally finite metric tree of degree $\geq 3$ embedded in the extended Bruhat–Tits building $\mathcal{B}_{\text{ext}}$.*

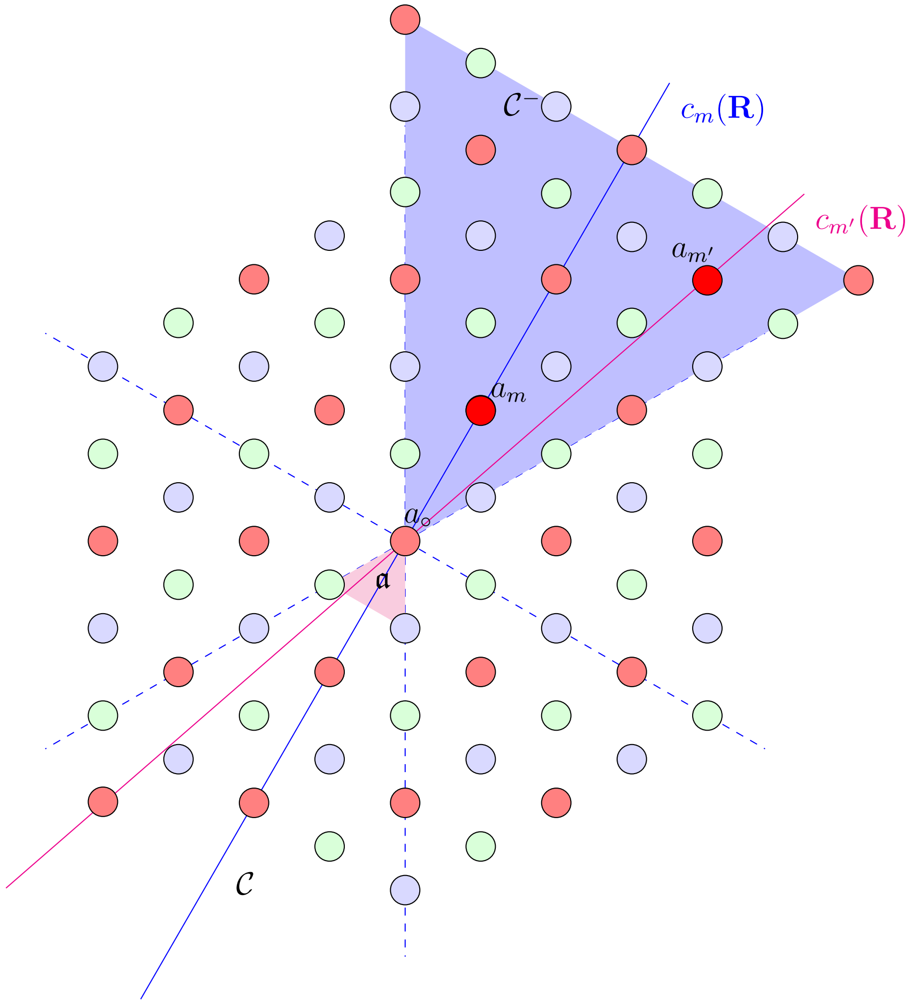
In the following figure, we consider the case when $\mathbf{G}$ is simply connected semi-simple, of split $F$-rank one, its extended Bruhat–Tits building is a tree and there is only one (modulo $M^1$) choice for $m \in M^{--}$ [11]: we highlight one single apartment $\mathcal{A}$, the red vertices represent the $G$-orbit of the special vertex $a_\circ$ and the edge in red is the alcove $\mathfrak{a} \subset \mathcal{A}$:




In the following figure, we consider the case when $G$ is semisimple adjoint type group of type $A_2$. We draw the apartment $\mathcal{A}$ and give two different possible choices $m, m'$ lying in $M^{--}$ inducing two special points $a_m$ and $a_{m'}$. As suggested by the figure below, an

[11] Let $\mathbf{G} = \mathbf{SL}_2$ over $\mathbb{Q}_p$, $\mathbf{B}$ be the Borel subgroup of upper triangular matrices and $\mathbf{S}$ be the split maximal torus of diagonals. We have $S^{--} = \{\text{diag}(p, p^{-1})\}$ up to units.

intuitive reformulation for the condition (2) of Definition V.3.1.1 on $m \in M^-$ would be to require that $a_m$ do not lie in the walls of the opposite vectorial chamber $\mathcal{C}^-$.




## V.3.2 Norm-compatible systems of vertices

Let $H$ be a group of isometries acting on the Bruhat–Tits building $\mathcal{B}(G)_{\text{ext}}$. Suppose there exists a $m \in M^{--}$ for which $H$ acts on the associated tree $\mathcal{T}_m$, i.e. $H \cdot \mathcal{T}_m = \mathcal{T}_m$. Recall that by construction, $\mathcal{T}_m$ comes with a distinguished half geodesic $c_m(]\infty, 0])$. We consider the following "Moufang" type hypothesis

$$\textbf{(Mfng)}: \text{H fixes the distinguished half line } c_m(]\infty,0]) \text{ and}$$
$$\text{acts transitively on the half lines opposite to it.}$$

**Example** V.3.2.1. *The group $\mathbf{SL}(2)(\mathbb{Z}_{(p)})$ (Localization, uncomplteted) fails to act transitively on the directions of the Bruhat–Tits building of $\mathbf{SL}(2)(\mathbb{Z}_p)$, but the stabilizer of a segment acts transitively on the vertices next to one of its end points, and this guarantees*

***(Mfng)***.

**Theorem** V.3.2.1. *Assume that **(Mfng)** holds for $H$ and $m \in M^{--}$. There exists then a sequence $\{a_{m^t}\}_{t\in\mathbb{N}}$ of special vertices in the fixed extended apartment $\mathcal{A}_{\text{ext}}$, such that*

$$\sum_{k=0}^{|W|} A_k \operatorname{Tr}_{t+k,t}(a_{m^{t+k}}) = 0, \quad \text{in } \mathbb{Z}[\mathcal{B}^\circ_{\text{ext}}]$$

*where $W = W(\mathbf{S})$ is the Weyl group, $A_k \in \mathcal{H}_K(R)$ some Hecke operators and $\operatorname{Tr}_{t+k,t} := \sum_{h\in \operatorname{Stab}_H(a_{m^t})/\operatorname{Stab}_H(a_{m^{t+k}})} h$.*

*Proof.* We have a left action

$$\mathcal{U}(m) \quad \times \quad \mathbb{Z}[\mathcal{T}_m \cap \mathcal{B}^\circ_{\text{ext}}] \longrightarrow \mathbb{Z}[\mathcal{T}_m \cap \mathcal{B}^\circ_{\text{ext}}]$$

where, $\mathcal{U}(m) := \langle \mathcal{U}_{m^t} : t \in \mathbb{N}\rangle$, and $\mathbb{Z}[\mathcal{T}_m \cap \mathcal{B}^\circ_{\text{ext}}]$ is the orbit of $a_1 = (a_\circ, 0) \in \mathcal{A}_{\text{ext}}$ under the semi-group $\{m^t u : t \in \mathbb{N}, u \in U^+\}$. Recall that for every $t \in \mathbb{N}$ we have

$$\mathcal{U}_m(a_{m^t}) = \sum_{u\in I^+/mI^+m^{-1}} m^t u \cdot a_m.$$

For every $t \in \mathbb{N}\setminus\{0\}$ and every $u \in I^+/mI^+m^{-1}$, the invariance of the metric with respect to $m^t$ then with respect to $u$ (which fixes $a_1$) imply

$$d(a_{m^t}, m^t u \cdot a_m) = d(a_1, u \cdot a_m) = d(a_1, a_m).$$

We shall normalize the distance so that $d(a_1, a_m) = 1$, thus

$$d(a_1, m^t u \cdot a_m) = d(a_1, \underbrace{m^t u m^{-t}}_{\in I^+} m^{t+1} \cdot a_1) = d(a_1, a_{m^{t+1}}) = d(a_1, a_{m^t}) + 1.$$

Set $\operatorname{Succ}(a_{m^t}) := \{b \in \mathcal{T}_m \cap \mathcal{B}^\circ_{\text{ext}} : d(b, a_1) = d(a_{m^t}, a_1) + 1\}$. We claim that

$$\mathcal{U}_m a_{m^t} = \sum_{\operatorname{Succ}(a_{m^t})} b.$$

The support of the term on the left is clearly contained in the successors on the right, we will show then the opposite containment. Let $b \in \operatorname{Succ}(a_{m^t})$, by our normalization $b$ must lie in $uc_m(\mathbb{Z})$ for some $u_b \in U^+$ and one has $u_b^{-1} \cdot b = a_{m^{t+1}}$. Now by (**Mnfg**), one can choose $u$ in the stabilizer $m^t \widetilde{K} m^{-t}$ hence

$$u_b \in m^t \widetilde{K} m^{-t} \cap U^+ = m^t(\widetilde{K} \cap U^+)m^{-t} = m^t I^+ m^{-t},$$

for the last equality, see proof of Proposition III.7.0.2. Accordingly $b \in m^t I^+ \cdot a_{m^1} \subset \operatorname{Support}(\mathcal{U}_m a_{m^t})$, which shows the claim. Define $H_t$ to be the stabilizer in $H$ of the geodesic segment $[a_1, a_{m^t}]$. The assumption (**Mnfg**) implies that for each $t \geq 1$, the subgroup $H_t$ acts transitively on the vertices appearing in the support of $\mathcal{U}_m a_{m^t}$ (i.e. acts transitively on the successors neighbours $\mathrm{Succ}(a_{m^t})$). Therefore, for any $t > 0$, one has

$$\mathcal{U}_m a_{m^t} = \sum_{\mathrm{Succ}(a_{m^t})} b = \sum_{h \in H_t/H_{t+1}} h \cdot a_{m^{t+1}}.$$

Let $Q_m(X) := \sum_{k=0}^{|W|} A_k X^k \in \mathcal{H}_K(R)[X]$ be the minimal polynomial annihilating the operator $\mathcal{U}_m$ (Theorem V.1.3.1). Therefore, we have for any $t \in \mathbb{N} \setminus \{0\}$[12]

$$\begin{aligned} 0 &= \sum_{k=0}^{|W|} A_k \circ \mathcal{U}_{m^k}(a_{m^t}) \\ &= \sum_{k=0}^{|W|} A_k \operatorname{Tr}_{t+k,t}(a_{m^{t+k}}). \end{aligned}$$ □

**Definition** V.3.2.1. *We will call such a sequence $\{a_{t^n}\}_{n\in\mathbb{N}}$ a $H$-norm compatible system of vertices in $\mathcal{B}_{\mathrm{ext}}$.*

**Remark** V.3.2.1. *Theorem V.3.2.1 above generalizes the series of lemmas proved in [BBJ18, §3.2] where $G = U(3) \times U(2)$ and $H = U(2)$. Without going into details: the choice of the translation $m$ with which we construct the norm compatible system of vertices in* loc. cit. *comes from a local cocharacter $\mu$ that is induced by the conjugacy class defining the Shimura datum: $m = \mu(\varpi^{-1})$. Now using §IV we see that the Hecke polynomial (the one provided by Blasius–Rogawski as in Definition IV.3.0.1) as computed in [Jet16, §4] is divisible by the minimal polynomial of the operator $\mathcal{U}_{\mu(\varpi^{-1})}$ (a slight variant of $U_V U_W$* in loc. cit., *compare this to [BBJ18, §3.2]). An almost immediate corollary of Theorem V.3.2.1 is the main theorem of [BBJ18, 1.2], where the relevant hypothesis (**Mnfg**) for this case is [Jet16, Lemma 3.6].*

[12] The degree of the polynomial $Q_m$ is $|W/W^m|$ where $W^m$ is the stabilizer of $m$ in the relative Weyl group $W$. Since $m \in M^{--}$, then by condition (2) (Definition V.3.1.1) we have $\deg(Q_m) = |Wm| = |W|$.

# CHAPTER VI

# UNITARY SHIMURA VARIETIES AND SPECIAL CYCLES

## VI.1 Notations

We fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and an embedding $i_{\overline{\mathbb{Q}}}\colon \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Let $F \subset \overline{\mathbb{Q}}$ be a totally real number field. Let $E$ be an imaginary quadratic extension of $F$, $E$ is usually said a CM field. Set $[E:\mathbb{Q}] = 2[F:\mathbb{Q}] = 2d$ and let

$$\Sigma_F := \mathrm{Hom}_{\mathbb{Q}}(F, \overline{\mathbb{Q}}) = \mathrm{Hom}_{\mathbb{Q}}(F, \mathbb{R}) = \{\, \iota_i \colon F \to \mathbb{R}, 1 \leq i \leq d\}$$

be the set of real embedding corresponding to the archimedean places of $F$. Let $\tau\colon x \mapsto x^{\tau}$ be the non-trivial element of $\mathrm{Gal}(E/F)$.

For each place $v$ of $F$ (possibly archimedean), we fix an algebraic closure $\overline{F}_v$ of $F_v$, and consider the set $\Sigma_{E,v} := \mathrm{Hom}_F(E, \overline{F}_v)$. The Galois group $\mathrm{Gal}(\overline{F}_v/F_v)$ acts on $\Sigma_{E,v}$ by post-composition: an element $\sigma \in \mathrm{Gal}(\overline{F}_v/F_v)$ sends any homomorphism $\varphi \in \Sigma_{E,v}$ to $\sigma \circ \varphi \in \Sigma_{E,v}$. This yields a bijection

$$\Sigma_{E,v}/\mathrm{Gal}(\overline{F}_v/F_v) \simeq \{w \text{ a place of } E\colon w \mid v\}$$

between $\mathrm{Gal}(\overline{F}_v/F_v)$-orbits of $F$-embeddings of $E$ in $\overline{F}_v$ and the places of $E$ above $v$. Set

$$\Sigma_E := \mathrm{Hom}_{\mathbb{Q}}(E, \overline{\mathbb{Q}}) = \mathrm{Hom}_{\mathbb{Q}}(E, \mathbb{C}) = \bigsqcup_{\iota \in \Sigma_F} \Sigma_{E,\iota}.$$

For each $1 \leq i \leq d$, choose an embedding $\widetilde{\iota}_i \in \Sigma_E$ that extends $\iota_i \in \Sigma_F$. The set $\widetilde{\Sigma}_E = \{\widetilde{\iota}_i \colon 1 \leq i \leq d\}$ is a CM type for $E$, because $\Sigma_E := \widetilde{\Sigma}_E \sqcup \widetilde{\Sigma}_E^{\tau}$. We denote by $c\colon x \mapsto \overline{x}$ the complex conjugation of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, we then have $c \circ \widetilde{\iota}_i = \widetilde{\iota}_i \circ \tau$.

We fix, for each finite place $v$ of $F$, an embedding $\iota_v \colon \overline{F} \hookrightarrow \overline{F}_v$. We may then view elements of $\Sigma_E$ (in particular $\Sigma_{E,\iota}$) as $v$-adic embeddings of $E$:

$$\Sigma_E \longrightarrow \Sigma_{E,v} \qquad \widetilde{\iota} \longmapsto \widetilde{\iota}_v := \iota_v \circ \widetilde{\iota}.$$

We fix a place of $E$ above each finite place of $F$ as follows: if $v$ splits in $E$, we fix the place $w_v$ to be the one defined by $\iota_v$ and by abuse of notation, denote the other place by $\overline{w}_v$. If $v$ is inert/ramified in $E$, we abuse notation and denote also by $v$ the unique place of $E$ above it.

## VI.2 The groups

Fix a positive integer $n > 1$ and let $(V, \psi)$ be a non-degenerate hermitian $E$-space of dimension $n+1$. Suppose that [1]

$$\mathrm{sign}((V,\psi) \otimes_{F,\iota_i} \mathbb{R}) = \begin{cases} (n,1) & \text{if } i = 1, \\ (n+1,0) & \text{if } i \neq 1. \end{cases}$$

We consider the $F$-algebraic reductive group of unitary isometries $\mathbf{U}(V,\psi)$, this is a connected reductive group whose $R$-points, for any $F$-algebra $R$, are given by

$$\mathbf{U}(V,\psi)(R) = \{g \in \mathbf{GL}(V \otimes_F R)\colon \psi(gx, gy) = \psi(x,y), \forall x, y \in V \otimes_F R\}.$$

We will be mainly interested in the cases where $R = F_v$, the completion of $F$ at finite places $v$, or $R = \mathbb{A}_{F,f}$, the ring of finite adeles of $F$. Since our hermitian form $\psi$ is fixed, we shall refer to the group of unitary isometries $\mathbf{U}(V,\psi)$ as $\mathbf{U}(V)$.

Let $v \in V$ be an anisotropic vector, every other $v' \in V$ such that $\psi(v',v') = \psi(v,v)$ is conjugate under $\mathbf{SU}(V)(F) = \mathbf{U}(V)(F) \cap \mathbf{SL}(V)(E)$ to $v$, (see [Shi08, 1.5]). Fix once and for all an anisotropic vector $w_{n+1} \in V$. Without loss of generality, assume that $\psi(w_{n+1}, w_{n+1}) = 1$ [2]. Set $D = E \cdot w_{n+1}$ and $W = D^{\perp}$. Hence, the signature of $(D, \psi|_D)$ is $(1,0)$ at all archimedean places of $F$. Consequently, the induced hermitian subspace $(W, \psi|_W)$ has signature $(n-1, 1)$ at the distinguished archimedean place $\iota_1$ and $(n, 0)$ at the other archimedean places $\iota_i, \forall 2 \leq i \leq n$. Similarly, we associate to $(W, \psi|_W)$ the $F$-algebraic group of unitary isometries $\mathbf{U}(W)$.

Set $\mathbf{G}_V := \mathrm{Res}_{F/\mathbb{Q}} \mathbf{U}(V)$ and $\mathbf{G}_W := \mathrm{Res}_{F/\mathbb{Q}} \mathbf{U}(W)$. Thus, for $\star \in \{V, W\}$ and for any

[1] For every $F$-algebra $R$, set $E_R := E \otimes_F R$ and $V_R := V \otimes_F R = V \otimes_E E_R$. We define the action of $\tau$ on $E_R = E \otimes_F R$ by letting it act on left component. Then, extend $\psi$ to a Hermitian form $\psi_R\colon V_R \times V_R \to E_R$ as follows

$$\psi_R(v \otimes x, v' \otimes y) = \psi_R(v, v') x y^{\tau}, \quad \forall v, v' \in V \text{ and } \forall x, y \in E_R.$$

For example: For each $1 \leq i \leq d$, the fixed embedding $\iota_i\colon F \to F_v$ induces a natural $F$-algebra structure on $\mathbb{R}$. Now letting $R = \mathbb{R}$, one gets $(V,\psi) \otimes_{F,\iota_i} \mathbb{R} := (V_R, \psi_R)$, the hermitian space $(V \otimes_{F,\iota_i} \mathbb{R}, \psi_i)$ over $E \otimes_{F,\iota_i} \mathbb{R}$.

[2] If $\psi(v,v) \neq 0$, one can choose any non-zero vector $v' \in E \cdot v$ and consider the modified hermitian $E$-space $(V, \frac{1}{\psi(v,v)}\psi)$. But, we may have changed the signature. A better argument: By density of $V$ in $V_{\iota_1}$, there is a vector $v \in V$ with $\psi(v,v)$ positive at $\iota_1$, hence everywhere. We choose this $v$ and consider the hermitian $E$-space $(V, \frac{1}{\psi(v,v)}\psi)$. Although the hermitian $E$-spaces are different, the associated unitary groups are isomorphic.

$\mathbb{R}$-algebra $R$, we have

$$\begin{aligned}\mathbf{G}_{\star,\mathbb{R}}(R) = \mathbf{U}(\star)(F \otimes_{\mathbb{Q}} R) &= \mathbf{U}(\star)(F \otimes_{\mathbb{Q}} \mathbb{R} \otimes_{\mathbb{R}} R) \\ &= \mathbf{U}(\star)(\bigoplus_{\iota \in \Sigma_F} \mathbb{R}_\iota \otimes_{\mathbb{R}} R) \\ &= \prod_{\iota \in \Sigma_F} \mathbf{U}(\star)(R_{\iota_i}),\end{aligned}$$

where $\mathbb{R}_\iota$ is just $\mathbb{R}$ indexed by elements of $\Sigma_F$, and $R_\iota$ is $R$ endowed with the $F$-algebra structure given by the embedding $\iota\colon F \hookrightarrow \mathbb{R}$. This yields $\mathbf{G}_{\star,\mathbb{R}} = \prod_{\iota\in\Sigma_F} \mathbf{G}_{\star,\iota}$ where,

$$\mathbf{G}_{\star,\iota} = \mathbf{U}((\star,\psi) \otimes_{F,\iota} \mathbb{R}) \simeq \begin{cases} \mathbf{U}(\dim_E \star - 1, 1)_{\mathbb{R}} & \text{if } \iota = \iota_1, \\ \mathbf{U}(\dim_E \star, 0)_{\mathbb{R}} & \text{if } \iota \neq \iota_1. \end{cases}$$

Likewise

$$\mathbf{G}_{\star,\mathbb{C}} = \prod_{\widetilde{\iota}\in\widetilde{\Sigma}_E} \mathbf{G}_{\star,\widetilde{\iota}} \quad \text{where} \quad \mathbf{G}_{\star,\widetilde{\iota}} = \mathbf{GL}(\star \otimes_{E,\widetilde{\iota}} \mathbb{C}) \simeq \mathbf{GL}(\dim_E \star)_{\mathbb{C}}.$$

By left-exactness of the Weil restriction we have an embedding of $\mathbb{Q}$-algebraic groups $\mathbf{G}_W \hookrightarrow \mathbf{G}_V$ that identifies $\mathbf{G}_W$ with the subgroup of $\mathbf{G}_V$ given by:

$$\{g \in \mathbf{G}_V(R) \subset \mathbf{GL}(V \otimes_{\mathbb{Q}} R)\colon g \cdot x = x, \quad \forall x \in D \otimes_{\mathbb{Q}} R\},$$

for any $\mathbb{Q}$-algebras $R$. Let $\mathbf{G} = \mathbf{G}_V \times \mathbf{G}_W$ and $\mathbf{H} = \Delta(\mathbf{G}_W) \subset \mathbf{G}$, where $\Delta$ denotes the diagonal embedding $\Delta\colon \mathbf{G}_W \hookrightarrow \mathbf{G}$.

## VI.3 The Deligne torus and variant

We refer the reader to §II.1.3.1 for more details.

- The torus $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_{m,\mathbb{C}}$ is called the Deligne torus and is usually denoted by $\mathbb{S}$. we will also consider its norm one subgroup $\mathbf{U}(1) := \mathbf{U}_{\mathbb{C}/\mathbb{R}}(1)$. As we have seen in §II.1.3.1, we get by base change to $\mathbb{C}$ a canonical isomorphism of $\mathbb{C}$-tori

  $$\mathbb{S}_{\mathbb{C}} \simeq \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}},$$

  where the factors are ordered in the way that $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times \to \mathbb{S}(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times$ is the map $z \mapsto (z, \overline{z})$.

  Define the cocharacter $\mu = \mu_{\mathbb{S}}\colon \mathbb{G}_{m,\mathbb{C}} \to \mathbb{S}_{\mathbb{C}}$ (introduced at the end of §II.1.3.1), given on $\mathbb{C}$-points by: $\mathbb{C}^\times \to \mathbb{S}_{\mathbb{C}}(\mathbb{C}) \simeq \mathbb{C}^\times \times \mathbb{C}^\times$, $z \mapsto (z, 1)$, one may also consider $\mu^c = \overline{\mu}_{\mathbb{S}}$ ($c$ for complex conjugation) given on $\mathbb{C}$-points by $z \mapsto (1, z)$.

- For each $\iota \in \Sigma_F$, consider the extension $E_\iota := E \otimes_{F,\iota} \mathbb{R}/\mathbb{R}$. We will use the notation

$\iota\mathbb{S} := \mathrm{Res}_{E_\iota/\mathbb{R}}\,\mathbb{G}_{m,E_\iota}$ and $\widetilde{\iota}\mathbf{U}(1) := \mathbf{U}_{E_\iota/\mathbb{R}}(1)$. Likewise, a base change to $E_\iota$ yields an isomorphism

$$\iota\mathbb{S}_{E_\iota} \simeq \mathbb{G}_{m,E_\iota} \times \mathbb{G}_{m,E_\iota},$$

where the factors are again ordered in the way that $E_\iota^\times \to \iota\mathbb{S}(E_\iota) = E_\iota^\times \times E_\iota^\times$ is the map $z \mapsto (z, z^\tau)$. Define similarly the cocharacter $\mu_\iota = \mu_{\iota\mathbb{S}} \colon \mathbb{G}_{m,E_\iota} \to \iota\mathbb{S}_{E_\iota}$, given on $E_\iota$-points by: $E_\iota^\times \to \iota\mathbb{S}_{E_\iota}(E_\iota) \simeq E_\iota^\times \times E_\iota^\times$, $z \mapsto (z, 1)$, one may also consider $\mu^\tau = \overline{\mu}_{\iota\mathbb{S}}$ given on $\mathbb{C}$-points by $z \mapsto (1, z)$.

- The distinguished complex embedding $\widetilde{\iota}_1 \colon E \to \mathbb{C}$, induces an isomorphism of fields

  $$\jmath_1 \colon E_{\iota_1} = E \otimes_{F,\iota_1} \mathbb{R} \xrightarrow{\ \simeq\ } \widetilde{\iota}_1(E) \otimes_{F,\iota_1} \mathbb{R} = \mathbb{C}\ ,$$

  and so yields an isomorphisms of $\mathbb{R}$-groups $\ \jmath_1 \colon \mathbb{S} \xrightarrow{\ \simeq\ } \iota_1\mathbb{S}\ ,\ \mathbf{U}(1) \xrightarrow{\ \simeq\ } \widetilde{\iota}\mathbf{U}(1)\ .$ Moreover, the base change of $\jmath_1$ to $E_{\iota_1}$ is compatible with the base change to $\mathbb{C}$, in the sense that

  $$\jmath_1 \circ \mu = \mu_{\iota_1} \text{ and } \jmath_1 \circ \mu^c = \mu_{\iota_1}^\tau,$$

  and the following diagram

  $$\begin{tikzcd} \mathbb{S} \arrow[r, two heads, "z\mapsto \frac{z}{\bar{z}}"] \arrow[d, "\jmath_1"', "\simeq"] & \mathbf{U}(1) \arrow[d, "\simeq"', "\jmath_1"] \\ \iota_1\mathbb{S} \arrow[r, two heads, "s\mapsto \frac{s}{s^\tau}"] & \widetilde{\iota}\mathbf{U}(1) \end{tikzcd} \tag{VI.1}$$

  commutes (see the end of §II.1.3.3).

# VI.4 The Hermitian symmetric domains

Let $\mathfrak{B}_V = (w_1, \cdots, w_{n+1})$ be an orthogonal $E$-basis of $(V, \psi)$, here $w_{n+1}$ is the $E$-generator fixed in §VI.2 of the $E$-line $D = W^\perp$. Thus, $\mathfrak{B}_W := (w_1, \cdots, w_n)$ is also an orthogonal $E$-basis of $(W, \psi)$. Recall that $E_{\iota_1} = E \otimes_{F,\iota_1} \mathbb{R} (\simeq \widetilde{\iota}_1(E) \otimes_{F,\iota_1} \mathbb{R} = \mathbb{C})$ and let $\mathfrak{B}_{V,1} = (w_{1,1}, \cdots, w_{n+1,1})$ be the orthogonal $E_{\iota_1}$-basis of $(V_{\iota_1} := V \otimes_{F,\iota_1} \mathbb{R}, \psi_1)$ obtained from $\mathfrak{B}_V$ by base change along the distinguished $\iota_1 \colon F \hookrightarrow \mathbb{R}$. Since the signature of $(V_{\iota_1}, \psi_1)$ is $(n, 1)$, we may and will assume that $c_1 = \psi_1(w_{1,1}, w_{1,1}) < 0$ and $c_i = \psi_1(w_{i,1}, w_{i,1}) > 0$ for all $1 < i \leq n+1$. Consider for simplicity the orthogonal basis $\mathfrak{B}'_{V,1} = \left(w'_{1,1}, w'_{2,1} \cdots, w'_{n+1,1}\right) = \left(\frac{1}{\sqrt{-c_1}} w_{1,1}, \frac{1}{\sqrt{c_2}} w_{2,1} \cdots, \frac{1}{\sqrt{c_{n+1}}} w_{n+1,1}\right)$, thus the Hermitian form $\psi_1$ with respect to $\mathfrak{B}'_{V,1}$ is given by

$$\psi_1(x, y) = -x_1 y_1^\tau + \cdots + x_n y_n^\tau + x_{n+1} y_{n+1}^\tau,$$

where $x = \sum_{i=1}^{n+1} x_i w'_{i,1}$ and $y = \sum_{i=1}^{n+1} y_i w'_{i,1}$ are two elements of $V_{\iota_1}$, with $x_i, y_i \in E_{\iota_1}$. This form corresponds to the matrix $J_{\mathfrak{B}'_V} = \operatorname{diag}(-1, 1, \cdots, 1)$. Likewise, let $J_{\mathfrak{B}'_W} = \operatorname{diag}(-1, 1, \cdots, 1)$ be the hermitian matrix corresponding to the basis $\mathfrak{B}'_W = (w'_{1,1}, w'_{2,1}, \cdots, w'_{n,1})$.

Let us view the $E_{\iota_1}$-vectors space $V_{\iota_1} \simeq E_{\iota_1}^{n+1} (\simeq \mathbb{C}^{n+1})$ as being a union

$$V_1^- \cup V_1^0 \cup V_1^+,$$

of negative (resp. null, positive) vectors $x \in V_{\iota_1}$, depending on the sign of $\psi_1(x, x) \in \mathbb{R}$. For instance

$$V_1^- = \left\{ x = \sum_{i=1}^{n+1} x_i w'_{i,1} \in V_{\iota_1} : \psi_1(x, x) = -|x_1|^2 + \sum_{i=2}^{n+1} |x_i|^2 < 0 \right\}.$$

For every non-zero $x \in V_{\iota_1}$, all non-zero vectors of the "complex"[3] line $E_{\iota_1} x$ have the same sign as $x$. Therefore, we can attach to each $E_{\iota_1}$-line in $V_{\iota_1}$ a sign in $\{-, 0, +\}$. In particular, we may think of $V_1^-$ as the union of all negative lines in $V_{\iota_1}$. Let $\mathcal{X}_V$ denote the set of negative lines in $V_1^-$.

Intersecting the negative lines with the hyperplane defined by

$$\left\{ x = \sum_{i=1}^{n+1} x_i w'_{i,1} \in V_{\iota_1} : x_1 = 1 \right\},$$

one gets an identification of $\mathcal{X}_V$ with the complex open ball of dimension $n$:

$$\mathbb{B}_n := \left\{ x = \sum_{i=1}^{n+1} x_i w'_{i,1} \in V_{\iota_1} : x_1 = 1 \text{ and } \sum_{i=2}^{n+1} |x_i|^2 < 1 \right\}.$$

For instance, the "complex" line $\ell_V = E_{\iota_1} w_{1,1}$ is an element of $\mathcal{X}_V$. The line $\ell_V$ corresponds to the centre $(1, 0, \cdots, 0)$ (for the basis $\mathfrak{B}'_V$) of the ball $\mathbb{B}_n$.

The group $\mathbf{G}_{V,\iota_1}(\mathbb{R}) \simeq \mathbf{U}(n,1)(\mathbb{R})$ acts transitively on the set $\mathcal{X}_V$ and the stabilizer of $\ell_V$ is isomorphic to $\mathbf{U}(n) \times \mathbf{U}(1)$ (See [Gol99, Lemma 3.1.3]).

The negative line $\ell_V$ determines a homomorphism of $\mathbb{R}$-algebraic groups $h_{\mathfrak{B}_V,1} \colon \mathbb{S} \longrightarrow \mathbf{G}_{V,\iota_1}$, as follows: the basis $\mathfrak{B}_V$ gives rise to the maximal $F$-subtorus of $\mathbf{U}(V)$

$$\mathbf{T}(\mathfrak{B}_V) := \mathbf{U}(Ew_1) \times \cdots \times \mathbf{U}(Ew_{n+1}) \subset \mathbf{U}_V,$$

similarly, the basis $\mathfrak{B}_{V,1}$ defines the maximal $\mathbb{R}$-subtorus of $\mathbf{G}_{V,\iota_1}$:

$$\mathbf{T}(\mathfrak{B}_{V,1}) := \mathbf{U}(\ell_V) \times \mathbf{U}(E_{\iota_1} w_{1,2}) \times \cdots \times \mathbf{U}(E_{\iota_1} w_{1,n+1}) = \mathbf{T}(\mathfrak{B}_V)_{\mathbb{R}} \subset \mathbf{G}_{V,\iota_1}.$$

The homomorphism of $E_{\iota_1}$-spaces $E_{\iota_1} \to \ell_V$ given by $z \mapsto z w_{1,1}$ induces an isomorphism of $\mathbb{R}$-groups $u_{V,1} \colon \widetilde{\iota}_1 \mathbf{U}(1)_{\mathbb{R}} \simeq \mathbf{U}(E_{\iota_1} w_{1,1})$. Extend $u_{V,1}$ to the morphism of $\mathbb{R}$-groups

[3] By a slight abuse of language, we use the adjective complex here, since this is really a complex line up to base change along $\widetilde{\iota}_1$.

$\widetilde{\iota}_1\mathbf{U}(1)_{\mathbb{R}} \to \mathbf{T}(\mathfrak{B}_{V,1}) \subset \mathbf{G}_{V,\iota_1}$ (by letting it act trivially on $\ell_V^{\perp} \subset V_{\iota_1}$), given with respect to the basis $\mathfrak{B}_{V,1}$, as follows:

$$u_{V,1}\colon \widetilde{\iota}_1\mathbf{U}(1)_{\mathbb{R}} \to \mathbf{G}_{V,\iota_1}, \quad z \mapsto \mathrm{diag}(z,1,\cdots,1).$$

Now, define the homomorphism $\widetilde{h}_{\mathfrak{B}_V,1}$ to be the composition of the following homomorphism of $\mathbb{R}$-algebraic groups:

$$\iota_1\mathbb{S} \xrightarrow{s\mapsto \frac{s}{s^\tau}} \widetilde{\iota}\mathbf{U}(1)_{\mathbb{R}} \xrightarrow{u_{V,1}} \mathbf{G}_{V,\iota_1}.$$

Accordingly, the desired $h_{\mathfrak{B}_V,1}\colon \mathbb{S} \longrightarrow \mathbf{G}_{V,\iota_1}$, is $h_{\mathfrak{B}_V,1} = \widetilde{h}_{\mathfrak{B}_V,1} \circ \jmath_1$, applying the square (VI.1) we also see that

$$h_{\mathfrak{B}_V,1}\colon z \longmapsto u_{V,1}\left(\frac{\jmath_1(z)}{(\jmath_1(z))^\tau}\right) = u_{V,1} \circ \jmath_1\left(\frac{z}{\overline{z}}\right).$$

The centralizer of $\widetilde{h}_{\mathfrak{B}_V,1}$ is the compact subgroup

$$\mathbf{U}(\ell_V) \times \mathbf{U}(\ell_V^{\perp}) \simeq \mathbf{U}(1)_{\mathbb{R}} \times \mathbf{U}(n,0)_{\mathbb{R}}.$$

Similarly, any negative line $\ell \in \mathcal{X}_V$ defines a homomorphism of $\mathbb{R}$-algebraic groups $\widetilde{h}_{V,\ell}\colon \widetilde{\iota}_1\mathbb{S} \to \mathbf{G}_{V,\iota_1}$, and the transitive action of $\mathbf{G}_{V,\iota_1}(\mathbb{R})$ on $\mathcal{X}_V$, there exists a unitary isometry $g \in \mathbf{G}_{V,\iota_1}(\mathbb{R})$ such that $g \cdot \ell_V = \ell$. Hence, the construction of $h_{V,1}$ above, yields the equality

$$\widetilde{h}_{V,\ell} = g\widetilde{h}_{V,1}g^{-1} \text{ and } h_{V,\ell} = gh_{V,1}g^{-1}.$$

Therefore, one may and will identify the set of negatives lines $\mathcal{X}_V$ with the $\mathbf{G}_{V,\iota_1}(\mathbb{R})$-conjugacy class of the homomorphism $h_{V,1}$.

The discussion above applies also to $W$. Let $\mathcal{X}_W$ be the set of negative definite $\mathbb{C}$-lines in $W_{\iota_1} := W \otimes_{F,\iota_1} \mathbb{R}(\simeq \mathbb{C}^n)$. The negative line $\ell_W = E_{\iota_1}w_{1,1}(= \ell_V)$, defines a homomorphism of $\mathbb{R}$-algebraic groups $\widetilde{h}_{W,1}\colon \widetilde{\iota}_1\mathbb{S} \to \mathbf{G}_{W,\iota_1}$, given on $\mathbb{R}$-points by[4]

$$z \in E_{\iota_1}^{\times} = \widetilde{\iota}_1\mathbb{S}(\mathbb{R}) \mapsto \mathrm{diag}(z/z^\tau,1,\cdots,1) \in \mathbf{G}_{W,\iota_1}(\mathbb{R}).$$

Consider the induced homomorphism $h_{W,1} = \widetilde{h}_{W,1} \circ \jmath_1\colon \mathbb{S} \longrightarrow \mathbf{G}_{W,\iota_1}$. Likewise, we identify as above $\mathcal{X}_W$ with the $\mathbf{G}_{W,\iota_1}(\mathbb{R})$-conjugacy class of the $\mathbb{R}$-algebraic homomorphism $h_{W,1}$.

For every $\star \in \{V,W\}$, the transitive action of $\mathbf{G}_{\star,\iota_1}(\mathbb{R})$ on $\mathcal{X}_\star$ naturally induces a transitive action of $\mathbf{G}_\star(\mathbb{R}) = \prod_{i=1}^d \mathbf{G}_{\star,\iota_i}(\mathbb{R})$ on $\mathcal{X}_\star$ with isotropy group the maximal connected compact subgroup:

$$\mathrm{Stab}_{\mathbf{G}_\star(\mathbb{R})}(h_\star) = (\mathbf{U}(\ell_\star^{\perp})\times\mathbf{U}(\ell_\star))\times\prod_{i=2}^d \mathbf{G}_{\star,\iota_i}(\mathbb{R}) \simeq \mathbf{U}(1)_{\mathbb{R}}\mathbf{U}(\dim_E\star-1)_{\mathbb{R}}\times\mathbf{U}(\dim_E\star,0)_{\mathbb{R}}^{d-1},$$

where, $h_\star\colon \mathbb{S} \to \mathbf{G}_{\star,\mathbb{R}}$ is the $\mathbb{R}$-algebraic homomorphism induced from $h_{\star,1}\colon \mathbb{S} \to \mathbf{G}_{\star,\iota_1}$, and

[4] With respect to the basis $\mathfrak{B}_{W,1} = (w_{1,1},\cdots,w_{n,1})$.

given on $\mathbb{R}$-points by

$$h_\star\colon z\mapsto (h_{\star,1}(z),\mathrm{Id}_2,\cdots,\mathrm{Id}_d).$$

We shall, therefore, identify $\mathcal{X}_\star$ with the $\mathbf{G}_\star(\mathbb{R})$-conjugacy class of the homomorphism $h_\star$. Let $\mathcal{X}=\mathcal{X}_V\times\mathcal{X}_W$, i.e. the $\mathbf{G}(\mathbb{R})$-conjugacy class of the homomorphism $h_V\times h_W\colon \mathbb{S}\to\mathbf{G}_\mathbb{R}$. The diagonal embedding $\Delta\colon W\hookrightarrow V\oplus W$ induces an embedding of $\mathcal{X}_W$ into $\mathcal{X}$. Write $\mathcal{Y}$ for $\Delta(\mathcal{X}_W)$; the $\mathbf{H}(\mathbb{R})$-conjugacy class of the homomorphism $\Delta(h_W)\colon\mathbb{S}\to\mathbf{H}_\mathbb{R}$. It can be easily checked that

$$\mathcal{Y}=\{h\in\mathcal{X}|h\colon\mathbb{S}\to\mathbf{G}_\mathbb{R}\text{ factors through }\Delta\colon\mathbf{H}_\mathbb{R}\hookrightarrow\mathbf{G}_\mathbb{R}\}.$$

## VI.5 The reflex field

We define

$$\mathbf{T}:=\mathrm{Res}_{E/\mathbb{Q}}\,\mathbb{G}_{m,E},\quad \mathbf{Z}:=\mathrm{Res}_{F/\mathbb{Q}}\,\mathbb{G}_{m,F},$$

and

$$\mathbf{T}^1:=\mathrm{Res}_{F/\mathbb{Q}}\,\mathbf{U}_{E/F}(1)=\ker(\mathrm{Norm}\colon\mathbf{T}\twoheadrightarrow\mathbf{Z}).$$

For $\star\in\{V,W\}$, let $\det:\mathbf{G}_\star\twoheadrightarrow\mathbf{T}^1$ be the determinant map. We have $\mathbf{G}_\star^{\mathrm{der}}=\ker(\det)=\mathrm{Res}_{F/\mathbb{Q}}\,\mathbf{SU}(\star)$.

Recall the maximal $F$-subtorus of $\mathbf{U}_\star$,

$$\mathbf{T}(\mathfrak{B}_\star)=\mathbf{U}(Ew_1)\times\cdots\times\mathbf{U}(Ew_{\dim\star})\simeq(\mathbf{U}_{E/F}(1))^{\dim\star}.$$

It induces the maximal $\mathbb{Q}$-subtorus of $\mathbf{G}_\star=\mathrm{Res}_{F/\mathbb{Q}}\,\mathbf{U}_\star$:

$$\mathbf{T}_{\mathfrak{B}_\star}:=\mathrm{Res}_{F/\mathbb{Q}}(\mathbf{T}(\mathfrak{B}_\star))=\mathrm{Res}_{F/\mathbb{Q}}\,\mathbf{U}(Ew_1)\times\cdots\times\mathrm{Res}_{F/\mathbb{Q}}\,\mathbf{U}(Ew_{\dim\star})\simeq\left(\mathbf{T}^1\right)^{\dim\star}.$$

Using the natural diagonal embedding $\mathbf{T}^1\to\mathbf{T}_{\mathfrak{B}_\star}$, we may view $\mathbf{T}^1$ as the center of the group $\mathbf{G}_\star$. Define $\mu_{\mathfrak{B}_\star}=(h_\star)_\mathbb{C}\circ\mu\in X_*(\mathbf{T}_{\mathfrak{B}_\star})_\mathbb{C}$ [5], where $\mu$ is the cocharacter of the Deligne torus introduced in §VI.3, and $(h_\star)_\mathbb{C}$ is, using the identification[6] $\mathbf{G}_{\star,\mathbb{C}}\simeq\prod_{\tilde\iota\in\tilde\Sigma_E}\mathbf{GL}(\star\otimes_{E,\tilde\iota}\mathbb{C})\simeq\mathbf{GL}(\dim_E\star)_\mathbb{C}$, given by

$$\begin{array}{rcl}(h_\star)_\mathbb{C}\colon\mathbb{S}_\mathbb{C}(\mathbb{C}) & \longrightarrow & \mathbf{T}_{\mathfrak{B}_{\star,\mathbb{C}}}(\mathbb{C})\\ (z_1,z_2) & \longmapsto & \left(\begin{pmatrix} z_1/z_2 & \\ & \mathrm{Id}_{\dim_E\star-1}\end{pmatrix},\mathrm{Id}_{\dim_E\star},\cdots,\mathrm{Id}_{\dim_E\star}\right).\end{array}$$

For every $?\in\{\mathbf{G},\mathbf{H},\mathbf{G}_V,\mathbf{G}_W\}$ and any field $\mathcal{K}\subset\overline{\mathbb{Q}}$, $\mathcal{M}_?(\mathcal{K})$ denotes the set of $?(\mathcal{K})$-

[5] We can recover $h_{\mathfrak{B}_\star}$ from $\mu_\star$ via $h_{\mathfrak{B}_\star}(z)=\mu_\star(z)\cdot\mu_\star(\overline{z})$.

[6] Determined by the choice of the CM type $\tilde\Sigma_1$.

conjugacy classes of (algebraic group) homomorphisms $\mathbb{G}_{m,\mathcal{K}} \to ?_{\mathcal{K}}$. Note that by construction, $h_\star\colon \mathbb{S} \to \mathbf{G}_\star$ factors through the torus $\mathbf{T}_{\mathfrak{B}_\star} \hookrightarrow \mathbf{G}_\star$.

By [Kot84a, (b) Lemma 1.1.3], we know that $\mathcal{M}_{\mathbf{G}_\star}(\mathbb{C}) = \mathcal{M}_{\mathbf{G}_\star}(\overline{\mathbb{Q}})$. Let $\mathcal{X}_{\star,\overline{\mathbb{Q}}} \in \mathcal{M}_{\mathbf{G}_\star}(\overline{\mathbb{Q}})$ be the class that corresponds to the $\mathbf{G}_\star(\mathbb{C})$-conjugacy class of $\mu_\star$ [7]. The reflex field $E(\mathbf{G}_\star, \mathcal{X}_\star)$ is defined to be the fixed field of the subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ fixing $\mathcal{X}_{\star,\overline{\mathbb{Q}}}$. We have (see proof of [Kot84a, Lemma 1.1.3])

$$\mathcal{M}_{\mathbf{G}_\star}(\overline{\mathbb{Q}}) \simeq X_*(\mathbf{T}_{\mathfrak{B}_\star})/W(\mathbf{G}_\star, \mathbf{T}_{\mathfrak{B}_\star})$$

where, $W(\mathbf{G}_\star, \mathbf{T}_{\mathfrak{B}_\star})$ denotes the absolute Weyl group of $\mathbf{T}_{\mathfrak{B}_\star}$ in $\mathbf{G}_\star$. Therefore, the reflex field $E(\mathbf{G}_\star, \mathcal{X}_\star)$ is also the field of definition of the $W(\mathbf{G}_\star, \mathbf{T}_{\mathfrak{B}_\star})$-orbit of $\mu_{\mathfrak{B}_\star}$. We now exhibit a rather explicit description of the $W(\mathbf{G}_\star, \mathbf{T}_{\mathfrak{B}_\star}) \rtimes \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module $X_*(\mathbf{T}_{\mathfrak{B}_\star})$ which will ease the computation of the reflex field. Recall that the duality between $X_*(\mathbf{T}_{\mathfrak{B}_\star})$ and $X^*(\mathbf{T}_{\mathfrak{B}_\star})$ is compatible with the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on both sides, in addition

$$\begin{aligned} X_*(\mathbf{T}_{\mathfrak{B}_\star}) &\simeq \mathrm{Hom}_{\mathbb{Z}-\mathrm{mod}}(X^*(\mathbf{T}_{\mathfrak{B}_\star}), \mathbb{Z}) \\ &= \bigoplus_{i=1}^{i=\dim \star} \mathrm{Hom}_{\mathbb{Z}-\mathrm{mod}}(X^*(\mathrm{Res}_{F/\mathbb{Q}}\, \mathbf{U}(Ew_i)), \mathbb{Z}). \end{aligned}$$

thus,

$$X_*(\mathbf{T}_{\mathfrak{B}_\star}) \simeq \bigoplus_{i=1}^{i=\dim \star} \mathrm{Hom}_{\mathbb{Z}-\mathrm{mod}}(X^*(\mathbf{T}^1)_i, \mathbb{Z}),$$

where, $X^*(\mathbf{T}^1)_i$ is just a copy of $X^*(\mathbf{T}^1)$ indexed by $1 \le i \le \dim_E \star$. On the other hand, we have

$$\mathbf{T}(\overline{\mathbb{Q}}) = \mathrm{Res}_{E/\mathbb{Q}}\, \mathbb{G}_{m,E}(\overline{\mathbb{Q}}) = (E \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})^\times = \bigoplus_{\iota \in \Sigma_K} (\overline{\mathbb{Q}}_\iota)^\times,$$

where $\overline{\mathbb{Q}}_\iota$ is just $\overline{\mathbb{Q}}$ indexed by elements of $\Sigma_E = \mathrm{Hom}_{\mathbb{Q}}(E, \overline{\mathbb{Q}})$ and endowed with the $E$-algebra structure given by the embedding $\iota\colon E \hookrightarrow \mathbb{R}$. Moreover, projections on each factor $(\overline{\mathbb{Q}}_\iota)^\times$ is an algebraic character that we denote by $f_\iota$. Hence, $\{f_\iota\colon \iota \in \Sigma_E\}$ is a $\overline{\mathbb{Q}}$-basis for $X^*(\mathbf{T})$. We have then a canonical isomorphism of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-modules

$$X^*(\mathbf{T})_{\overline{\mathbb{Q}}} \simeq \bigoplus_{\iota \in \Sigma_E} \mathbb{Z} f_\iota,$$

with the canonical Galois module on the right hand side. Define $\{f_\iota^\vee\colon \iota \in \Sigma_E\}$ to be the dual basis of $\{f_\iota\colon \iota \in \Sigma_E\}$ in $\mathrm{Hom}_{\mathbb{Z}-\mathrm{mod}}(X^*(\mathbf{T}), \mathbb{Z})$. A homomorphism $f \in \mathrm{Hom}_{\mathbb{Z}-\mathrm{mod}}(X^*(\mathbf{T}), \mathbb{Z})$ is completely determined by the $\mathbb{Z}$-values it attaches to the basis $\{f_\iota^\vee\colon \iota \in \Sigma_E\}$ or equivalently, to $\{\iota \in \Sigma_E\}$. Therefore, we obtain the isomorphism

$$\{f\colon \Sigma_E \to \mathbb{Z}\} \xrightarrow{\simeq} X_*(\mathbf{T})_{\overline{\mathbb{Q}}} \quad (\simeq \mathrm{Hom}_{\mathbb{Z}-\mathrm{mod}}(X^*(\mathbf{T})_{\overline{\mathbb{Q}}}, \mathbb{Z})),$$

[7]The $\mathbf{G}_\star(\mathbb{C})$-conjugacy class of $\mu_\star$ does not depend on the representatives $h_{\mathfrak{B}_\star} \in \mathcal{X}_\star$.

given by

$$f \longmapsto \left( \lambda_f \colon \overline{\mathbb{Q}} \to \mathbf{T}(\overline{\mathbb{Q}}) = \bigoplus_{\iota\in\Sigma_E} (\overline{\mathbb{Q}}_\iota)^\times, \quad z \mapsto \prod_{\iota\in\Sigma_E} f_\iota^\vee(z)^{f(\iota)} \right).$$

The embedding $\mathbf{T}^1 \hookrightarrow \mathbf{T}$ induces an injection $X_*(\mathbf{T}^1) \hookrightarrow X_*(\mathbf{T})$. We will try here to describe the submodule of $\{f \colon \Sigma_E \to \mathbb{Z}\}$ corresponding to $X_*(\mathbf{T}^1)$. We begin by describing the $\overline{\mathbb{Q}}$-points of $\mathbf{T}^1$ as follows (See §II.1.3.2)

$$\begin{aligned} \mathbf{T}^1(\overline{\mathbb{Q}}) &= \left\{ z \in \mathbf{T}(\overline{\mathbb{Q}}) \colon \chi(z) = 1, \forall \chi \in X^*(\mathbf{T})^{\mathrm{Gal}(E/F)} \right\} \\ &= \left\{ z \in \bigoplus_{\iota\in\Sigma_E} (\overline{\mathbb{Q}}_\iota)^\times \colon f_\iota(z) f_{\iota^\tau}(z) = 1, \ \forall \iota \in \Sigma_E \right\}. \end{aligned}$$

Therefore, one can identify $X_*(\mathbf{T}^1)$ with $\{f \colon \Sigma_E \to \mathbb{Z} | f(\iota) + f(\iota^\tau) = 0, \forall \iota \in \Sigma_E\}$. Recall, that for each $1 \leq i \leq d$, we have fixed a $\widetilde{\iota}_i \in \Sigma_E$ extending the fixed $\iota_i \in \Sigma_F$.

In conclusion, we have a isomorphism of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-modules between $X_*(\mathbf{T}^1)$ and $\mathrm{Hom}(\widetilde{\Sigma}_E, \mathbb{Z})$, and thus an isomorphism

$$\mathrm{Hom}(\widetilde{\Sigma}_E, \mathbb{Z}^{\dim\star}) \xrightarrow{\ \simeq\ } X_*(\mathbf{T}_{\mathfrak{B}_\star}).$$

The representative $\mu_{\mathfrak{B}_\star} \in X_*(\mathbf{T}_{\mathfrak{B}_\star})$ of the class $\mathcal{X}_{\star,\overline{\mathbb{Q}}}$ corresponds, under the isomorphism above, to the function

$$f_\star \colon \widetilde{\Sigma}_E \to \mathbb{Z}^{\dim\star}, \quad \widetilde{\iota}_i \mapsto \begin{cases} (1, 0, \cdots, 0) & \text{if } i = 1, \\ (0, \cdots, 0) & \text{if } 2 \leq i \leq d, \end{cases}$$

or, equivalently

$$f_\star \colon \Sigma_E \to \mathbb{Z}^{\dim\star}, \quad \widetilde{\iota} \mapsto \begin{cases} (1, 0, \cdots, 0) & \text{if } \widetilde{\iota} = \widetilde{\iota}_1, \\ (-1, 0, \cdots, 0) & \text{if } \widetilde{\iota} = \widetilde{\iota}_1{}^\tau, \\ (0, \cdots, 0) & \text{if } \widetilde{\iota} \notin \{\widetilde{\iota}_1, \widetilde{\iota}_1{}^\tau\}. \end{cases}$$

Since all considered tori split over $E$, the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ will then act on these objects by its projection on $\mathrm{Gal}(E/\mathbb{Q})$. The absolute Weyl group $W(\mathbf{G}_\star, \mathbf{T}_{\mathfrak{B}_\star}) \simeq \mathcal{S}^d_{\dim\star}$ acts on $\mathrm{Hom}(\Sigma_E, \mathbb{Z}^{\dim\star})$ by permuting the components in $\mathbb{Z}^{\dim\star}$, e.g. $W(\mathbf{G}_\star, \mathbf{T}_{\mathfrak{B}_\star}) f_\star = \{f_{\star,k} \colon \widetilde{\Sigma}_E \to \mathbb{Z}^{\dim\star}, \text{ with } 1 \leq k \leq \dim\star\}$ where

$$f_{\star,k} \colon \widetilde{\Sigma}_E \to \mathbb{Z}^{\dim\star}, \widetilde{\iota}_i \mapsto \begin{cases} (0, \cdots, 0, \underbrace{1}_{k^{\text{th}} \text{ position}}, 0, \cdots, 0) & \text{if } i = 1, \\ (0, \cdots, 0) & \text{if } 2 \leq i \leq d, \end{cases}.$$

Consequently, a Galois element fixing the Weyl orbit $W(\mathbf{G}_\star, \mathbf{T}_{\mathfrak{B}_\star}) \cdot f_\star$ must fix the $\iota_1$ component, hence is contained in $\mathrm{Gal}(\overline{\mathbb{Q}}/\iota_1(F))$. Now since, $f_\star^\tau \neq f_\star$, the Galois subgroup that fixes $W(\mathbf{G}_\star, \mathbf{T}_{\mathfrak{B}_\star}) \cdot f_\star$ is precisely $\mathrm{Gal}(\overline{\mathbb{Q}}/\widetilde{\iota}_1(E))$ and the field of definition of

$W(\mathbf{G}_\star, \mathbf{T}_{\mathfrak{B}_\star}) \cdot f_\star$ is

$$E(\mathbf{G}_\star, \mathcal{X}_\star) = \widetilde{\iota}_1(E).$$

In conclusion, we also have $E(\mathbf{H}, \mathcal{Y}) = E(\mathbf{G}, \mathcal{X}) = \widetilde{\iota}_1(E)$.

## VI.6 Reflex norm maps

For $\star \in \{V, W\}$, the discussion of §VI.5 shows that $\mu_{\mathfrak{B}_\star} = (h_{\mathfrak{B}_\star})_{\mathbb{C}} \circ \mu \in X_*(\mathbf{T}_\star)_{\mathbb{C}}$ is actually defined over $\widetilde{\iota}_1(E) = \widetilde{\iota}_1^{\tau}(E)$. Put $\iota_1\mathbf{T} := \mathrm{Res}_{\widetilde{\iota}_1(E)/\mathbb{Q}}\, \mathbb{G}_{m,\widetilde{\iota}_1(E)}$ and define the reflex norm map $r(\mu_{\mathfrak{B}_\star})$ to be the composition

$$\mathbf{T} \xrightarrow[\simeq]{\jmath_1} \iota_1\mathbf{T} \xrightarrow{\mathrm{Res}_{\widetilde{\iota}_1(E)/\mathbb{Q}}(\mu_{\mathfrak{B}_\star})} \mathrm{Res}_{\widetilde{\iota}_1(E)/\mathbb{Q}}(\mathbf{T}_{\mathfrak{B}_\star})_{\iota_1(F)} \xrightarrow{\mathrm{Norm}} \mathbf{T}_{\mathfrak{B}_\star} \xrightarrow{\det} \mathbf{T}^1$$

Therefore, $r(\mu_{\mathfrak{B}_\star}) = \mathrm{Res}_{F/\mathbb{Q}}\, \mathbf{N}_{\mathbf{U}_{E/F}(1)} \circ \jmath_1$, where $\mathbf{N}_{\mathbf{U}_{E/F}(1)}$ is (see §II.1.3.3)

$$\mathrm{Res}_{\widetilde{\iota}_1(E)/F}\, \mathbb{G}_{m,\widetilde{\iota}_1(E)}(R) \longrightarrow \mathbf{U}_{E/F}(1)(R), \qquad s \longmapsto \tfrac{s}{s^{\tau}}.$$

for any $F$-algebra $R$. This shows, in particular, that $r(\mu_{\mathfrak{B}_\star})$ is independent of the choice of the fixed basis $\mathfrak{B}_\star$, from now on we will denote this map by $\nu := r(\mu_{\mathfrak{B}_\star})$.

In addition, by the exact sequence (II.2) we obtain again[8] an exact sequence of $\mathbb{Q}$-tori

$$1 \longrightarrow \mathbf{Z} \longrightarrow \mathbf{T} \xrightarrow{\nu} \mathbf{T}^1 \longrightarrow 1. \tag{VI.2}$$

The above discussion justifies the possibility of omitting the distinguished embeddings $\widetilde{\iota}_1$ and $\iota_1$. Accordingly, we identify the abstract number fields $F$ with $\iota_1(F) \subset \mathbb{R}$ and $E$ with $\widetilde{\iota}_1(E) = \widetilde{\iota}_1^{\tau}(E) \subset \mathbb{C}$.

## VI.7 The Shimura varieties

Let us begin by proving some properties of the pairs $(\mathbf{G}, \mathcal{X})$ and $(\mathbf{H}, \mathcal{Y})$:

**Proposition** VI.7.0.1. *The pairs $(\mathbf{G}_\star, \mathcal{X}_\star)$, $\star \in \{V, W\}$ are Shimura data (See [Mil17b, Definition 5.5]), that is*

*SV1 The only characters of the induced representation $Ad \circ h_{\mathfrak{B}_\star} \colon \mathbb{S} \to \mathbf{GL}(\mathrm{Lie}(\mathbf{G}_{\star,\mathbb{C}}))$, are $z \mapsto z/\overline{z}, 1, \overline{z}/z$.*

*SV2 $Ad \circ h$ is a Cartan involution of $\mathbf{G}^{\mathrm{der}}_{\star,\mathbb{R}}$ for all $h \in \mathcal{X}_\star$.*

[8] On the one hand, the Weil restriction is left exact on the category of $\mathbb{Q}$-groups. On the other hand, the right surjectivity is a consequence of [CGP15, Corollary A.5.4(1)].

*SV3* $\mathbf{G}_\star^{\mathrm{ad}}$ *does not have any direct* $\mathbb{Q}$*-factor* $\mathbf{L}$ *on which* $h_{\mathfrak{B}_\star}$ *is trivial.*

*Proof.* For $(\mathbf{G}_\star, \mathcal{X}_\star)$, $\star \in \{V, W\}$:

SV1 Recall that we have defined the $\mathbb{R}$-algebraic homomorphisms $h_{\mathfrak{B}_\star}\colon \mathbb{S} \to \mathbf{G}_{\star,\mathbb{R}} \simeq \mathbf{U}(\dim_E \star - 1, 1)_\mathbb{R} \times \mathbf{U}(\dim_E \star)_\mathbb{R}^{d-1}$, given by

$$h_{\mathfrak{B}_\star}\colon \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times \ni z \mapsto \left(\begin{pmatrix} z/\overline{z} & \\ & 1_{\dim_E \star - 1}\end{pmatrix}, 1_{\dim_E \star}, \cdots, 1_{\dim_E \star}\right)$$

with respect to the basis $\mathfrak{B}_\star$. It is then, straightforward to see that the only characters of the induced representation $\mathrm{Ad} \circ h_{\mathfrak{B}_\star}\colon \mathbb{S} \to \mathbf{GL}(\mathrm{Lie}(\mathbf{G}_{\star,\mathbb{C}}))$, are $z \mapsto z/\overline{z}, 1, \overline{z}/z$.

SV2 The ss that $\mathrm{Ad} \circ h$ is a Cartan involution of $\mathbf{G}_{\star,\mathbb{R}}^{\mathrm{der}}$ for all $h \in \mathcal{X}_\star$. Since, all Cartan involutions are conjugate by an inner automorphism, we just need to verify this axiom for $h = h_{\mathfrak{B}_\star}$, that is to show that the following Lie group

$$\begin{aligned}\mathbf{G}_{\star,\mathbb{R}}^{\mathrm{ad}(h_{\mathfrak{B}_\star}(i))}(\mathbb{R}) &:= \{g \in \mathbf{SU}(\star)(\mathbb{C})\colon g h_{\mathfrak{B}_\star}(i) = h_{\mathfrak{B}_\star}(i)\overline{g}\} \\ &= \{g \in \mathbf{SU}(\star)_{F,\iota_1}(\mathbb{C})\colon g h_{\mathfrak{B}_\star}(i) = h_{\mathfrak{B}_\star}(i)\overline{g}\} \times \prod_{i=2}^{i=d} \mathbf{SU}(\star)_{F,\iota_i}(\mathbb{C}), \\ &\simeq \{g \in \mathbf{SL}(\star_{F,\iota_1} \otimes \mathbb{C})\colon g J_{\mathfrak{B}_\star} {}^t\overline{g} = J_{\mathfrak{B}_\star}, g h_{\mathfrak{B}_\star}(i) = h_{\mathfrak{B}_\star}(i)\overline{g}\} \times \mathbf{SU}(\dim_E \star)(\mathbb{C})^{d-1}\end{aligned}$$

is compact. Indeed, for all $2 \le i \le d$, the subgroup $\mathbf{SU}(\star)_{F,\iota}(\mathbb{C}) \simeq \mathbf{SU}(\dim_E \star)(\mathbb{C})$ is compact. While, for $\iota_1$, the fact that $h_{\mathfrak{B}_\star}(i) = J_{\mathfrak{B}_\star}$ shows[9] $\{g \in \mathbf{SL}(\star_{F,\iota_1} \otimes \mathbb{C})\colon g J_{\mathfrak{B}_\star} {}^t\overline{g} = J_{\mathfrak{B}_\star}$ and $g h_{\mathfrak{B}_\star}(i) = h_{\mathfrak{B}_\star}(i)\overline{g}\}$ is equal to

$$\mathbf{SU}(\star_{F,\iota_1})(\mathbb{C}) \cap \mathbf{O}(\star_{F,\iota_1} \otimes \mathbb{C}),$$

and this latter is a closed subgroup of the compact orthogonal group $\mathbf{O}(\star_{F,\iota_1} \otimes \mathbb{C})$. This proves the claim.

SV3 The third axiom requires that $\mathbf{G}_\star^{\mathrm{ad}}$ does not have any direct $\mathbb{Q}$-factor $\mathbf{L}$ on which $h_{\mathfrak{B}_\star}$ is trivial. By [Mil17b, Remark 4.6.], this holds if and only if $\mathbf{G}_\star^{\mathrm{ad}}$ is of noncompact type [Mil17b, Definition 3.18]. Consider the isogeny $\mathbf{G}_\star^{\mathrm{der}} = \mathbf{SU}(\star) \to \mathbf{G}_\star^{\mathrm{ad}}$, clearly $\mathbf{SU}(\star)(\mathbb{R})$ is noncompact due to the signature of $(\star_{F,\iota_1}, \psi_1)$ which is $(\dim_E \star - 1, 1)$. This proves $\mathbf{G}_\star^{\mathrm{ad}}$ is of noncompact type. □

The proposition above, shows that

$$(\mathbf{G}, \mathcal{X}) = (\mathbf{G}_V, \mathcal{X}_V) \times (\mathbf{G}_W, \mathcal{X}_W) \text{ and } (\mathbf{H}, \mathcal{Y}) = \Delta(\mathbf{G}_W, \mathcal{X}_W)$$

are also Shimura data, and implies, in particular, that the connected[10] spaces $\mathcal{X}_V, \mathcal{X}_W, \mathcal{X}$

[9] The two conditions above, gives the equality $g J_{\mathfrak{B}_\star} {}^t\overline{g} = J_{\mathfrak{B}_\star} \overline{g}^t \overline{g} = J_{\mathfrak{B}_\star}$, hence $g^t g = \mathrm{Id}_{\star_{\iota_1,\mathbb{R}} \otimes \mathbb{C}}$.

[10] The identification we have seen in §VI.4 between $\mathcal{X}_V$ and $\mathcal{X}_W$ and complex open balls shows that these spaces are connected and consequently $\mathcal{X}$ and $\mathcal{Y}$ are connected too.

and $\mathcal{Y}$ are Hermitian symmetric domains.

**Remark** VI.7.0.1. *Let* $\mathbf{X} \in \{V, W\}$*, then the Shimura datum* $(\mathbf{G}_\star, \mathcal{X}_\star)$ *verifies the following additional axioms* [Mil17b, Additional axioms, p. 63]*:*

*SV4* *The weight homomorphism*

$$w_{\mathcal{X}_\star} \colon \mathbb{G}_{m,\mathbb{R}} \xrightarrow{r \mapsto r^{-1}} \mathbb{S} \xrightarrow{h_{\mathfrak{B}_V}} \mathbf{G}_{\star,\mathbb{R}}$$

*is clearly trivial*[11]*, since* $h_{\mathfrak{B}_V}$ *factors through* $\mathbf{U}(1)$*, thus it is tautologically defined over* $\mathbb{Q}$*.*

*SV5* *The rational points of the center* $\mathbf{T}^1(\mathbb{Q})$ *are discrete in* $\mathbf{T}^1(\mathbb{A}_f)$*. Indeed,* $\mathbf{T}^1$ *is anisotropic over* $\mathbb{Q}$ *and remains anisotropic over* $\mathbb{R}$ *too, thus the set of real points* $\mathbf{T}^1(\mathbb{R})$ *must be compact which shows that* $\mathbf{T}^1(\mathbb{Q})$ *is discrete in* $\mathbf{T}^1(\mathbb{A}_f)$ [Mil17b, §5 - Arithmetic subgroups of tori]*.*

*SV6* *By definition, the center* $\mathbf{T}^1$ *splits over the CM-field* $E$*.* □

For every compact subgroup $K_{\mathbf{G}} \subset \mathbf{G}(\mathbb{A}_f)$ (resp. $K_{\mathbf{H}} \subset \mathbf{G}(\mathbb{A}_f)$), the Shimura variety $\mathrm{Sh}_{K_{\mathbf{G}}}(\mathbf{G}, \mathcal{X})$ (resp. $\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, \mathcal{Y})$) is the complex analytic space

$$\mathbf{G}(\mathbb{Q}) \backslash (\mathcal{X} \times (\mathbf{G}(\mathbb{A}_f)/K_{\mathbf{G}})) \quad (\text{resp. } \mathbf{H}(\mathbb{Q}) \backslash (\mathcal{Y} \times (\mathbf{H}(\mathbb{A}_f)/K_{\mathbf{H}}))),$$

where $\mathbf{G}(\mathbb{Q})$ (resp. $\mathbf{H}(\mathbb{Q})$) acts diagonally on $\mathcal{X} \times (\mathbf{G}(\mathbb{A}_f)/K_{\mathbf{G}})$ (resp. $\mathcal{Y} \times (\mathbf{H}(\mathbb{A}_f)/K_{\mathbf{H}})$).

**Proposition** VI.7.0.2. *The Shimura variety* $\mathrm{Sh}_{K_{\mathbf{G}}}(\mathbf{G}, \mathcal{X})$ *(resp.* $\mathrm{Sh}_{K_{\mathbf{G}}}(\mathbf{H}, \mathcal{Y})$*) has the following decomposition*

$$\bigsqcup_{g \in \mathcal{C}_{\mathbf{G}}} \Gamma_g \backslash \mathcal{X} \quad (\textit{resp.} \quad \bigsqcup_{h \in \mathcal{C}_{\mathbf{H}}} \Gamma_h \backslash \mathcal{Y}),$$

*where for* $\boldsymbol{X} \in \{\mathbf{G}, \mathbf{H}\}$*, we have used* $\mathcal{C}_{\boldsymbol{X}}$ *to denote the finite class group* $\boldsymbol{X}(\mathbb{Q}) \backslash \boldsymbol{X}(\mathbb{A}_f)/K_{\boldsymbol{X}}$ *and for* $x \in \mathcal{C}_{\boldsymbol{X}}$ *we put* $\Gamma_x := xK_{\boldsymbol{X}}x^{-1} \cap \boldsymbol{X}(\mathbb{Q}) \subset \boldsymbol{X}(\mathbb{R})$*.*

*Proof.* This is an application of [Mil17b, Lemma 5.13.]. □

The subgroups $\Gamma_x$ for $x \in \mathcal{C}_{\mathbf{G}}$ (resp. $x \in \mathcal{C}_{\mathbf{H}}$) are congruence[12] arithmetic subgroups of $\mathbf{G}(\mathbb{Q})$ (resp. $\mathbf{H}(\mathbb{Q})$) (See [Mil17b, Proposition 4.1]).

[11] As for any "connected" Shimura variety.

[12] We have an embedding $\pi \colon \mathbf{G} \hookrightarrow \mathbf{GL}(V \oplus W) \simeq \mathbf{GL}_{2n+1}$. A congruence subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is a subgroup containing a finite index subgroup of the form

$$\Gamma(N) = \pi(\mathbf{G})(\mathbb{Q}) \cap \{g \in \mathbf{GL}_{2n+1}(\mathbb{Z}) \colon g \equiv I_{2n+1} \mod N\},$$

for some integer $N$. The subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is arithmetic if $\pi(\Gamma)$ is commensurable with $\pi(\mathbf{G})(\mathbb{Q}) \cap \mathbf{GL}_{2n+1}(\mathbb{Z})$. Arithmetic and congruence subgroups of $\mathbf{H}(\mathbb{Q})$, are defined similarly.

By the work of Baily and Borel [BB66], each connected component $\Gamma_g\backslash\mathcal{X}$, $(g \in \mathcal{C}_{\mathbf{G}})$ (resp. $\Gamma_h\backslash\mathcal{Y}$, $(h \in \mathcal{C}_{\mathbf{H}})$) can be endowed with a natural structure of a complex quasi-projective variety, hence also $\mathrm{Sh}_{K_{\mathbf{G}}}(\mathbf{G},\mathcal{X})$ and $\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H},\mathcal{Y})$. Moreover, if $\Gamma_x$ (for $x$ in $\mathcal{C}_{\mathbf{G}}$ or $\mathcal{C}_{\mathbf{H}}$) is small enough (for example, if it is torsion-free) then $\Gamma_g\backslash\mathcal{X}$ is smooth and its algebraic structure is unique. In this thesis, we will be considering a stronger condition on the compact open subgroups $K_{\mathbf{G}} \subset \mathbf{G}(\mathbb{A}_f)$ and $K_{\mathbf{H}} \subset \mathbf{H}(\mathbb{A}_f)$, namely being neat [Pin89, §0.1], it prevents $\mathbf{G}(\mathbb{Q})$ (resp. $\mathbf{H}(\mathbb{Q})$) from having fixed point in $\mathcal{X} \times \mathbf{G}(\mathbb{A}_f)/K$ (resp. $(\mathcal{Y} \times \mathbf{H}(\mathbb{A}_f)/K_{\mathbf{H}})$), in which case $\mathrm{Sh}_{K_{\mathbf{G}}}(\mathbf{G},\mathcal{X})$ and $\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H},\mathcal{Y})$ are smooth.

We have a better understanding of the algebraic structure governing the Shimura varieties $\mathrm{Sh}_{K_{\mathbf{G}}}(\mathbf{G},\mathcal{X})$ and $\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H},\mathcal{Y})$. Indeed, It follows from results of Deligne, Borovoi, Milne and Moonen [Del71, Del79, Bor84, Mil83, Mil99, Moo98b], that every Shimura varietiy has a canonical model [13] defined over its reflex field.

From now on, we will reserve the notation $\mathrm{Sh}_{K_{\mathbf{G}}}(\mathbf{G},\mathcal{X})$ and $\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H},\mathcal{Y})$ for the corresponding canonical model over the reflex field $E(\mathbf{G},\mathcal{X}) = E(\mathbf{H},\mathcal{Y}) = E$. Therefore, the initial definition given in terms of double cosets gives actually the Complex points of these models:

$$\mathrm{Sh}_{K_{\mathbf{G}}}(\mathbf{G},\mathcal{X})(\mathbb{C}) = \mathbf{G}(\mathbb{Q})\backslash(\mathcal{X}\times(\mathbf{G}(\mathbb{A}_f)/K_{\mathbf{G}})), \quad \mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H},\mathcal{Y})(\mathbb{C}) = \mathbf{H}(\mathbb{Q})\backslash(\mathcal{Y}\times(\mathbf{H}(\mathbb{A}_f)/K_{\mathbf{H}})).$$

## VI.8 The projective system and Hecke action

For any two neat compact open subgroups $K' \subset K$, we have an obvious quotient map

$$\pi_{K',K}\colon \mathrm{Sh}_{K'}(\mathbf{G},\mathcal{X})(\mathbb{C}) \to \mathrm{Sh}_K(\mathbf{G},\mathcal{X})(\mathbb{C}),$$

which defines a finite étale morphism between $\mathrm{Sh}_{K'}(\mathbf{G},\mathcal{X}) \to \mathrm{Sh}_K(\mathbf{G},\mathcal{X})$. Taking the projective system over neat compact subgroups,

$$\mathrm{Sh}(\mathbf{G},\mathcal{X}) = \varprojlim_{K \text{ neat}} \mathrm{Sh}_K(\mathbf{G},\mathcal{X}),$$

we obtain a a scheme over $\mathbb{C}$, endowed with a continuous action of $\mathbf{G}(\mathbb{A}_f)$ (see [Del79, 2.1.4 and 2.7] and [Mil90, II.2 and II.10]). Define also the quotient map $\pi_{\mathbf{G},K}\colon \mathrm{Sh}(\mathbf{G},\mathcal{X}) \longrightarrow \mathrm{Sh}(\mathbf{G},\mathcal{X})/K = \mathrm{Sh}_K(\mathbf{G},\mathcal{X})$. The action of $g \in \mathbf{G}(\mathbb{A}_f)$ denoted $T_g\colon \mathrm{Sh}(\mathbf{G},\mathcal{X}) \to \mathrm{Sh}(\mathbf{G},\mathcal{X})$ on the system $(\mathrm{Sh}_K(\mathbf{G},\mathcal{X}))_{K \text{ neat}}$ defines an isomorphism of algebraic varieties between

[13] This canonical model depends only on the associated Shimura data. In particular, canonical models do not depend on the chosen open compact.

$\mathrm{Sh}_K(\mathbf{G},\mathcal{X})$ and $\mathrm{Sh}_{g^{-1}Kg}(\mathbf{G},\mathcal{X})$ (See [Mil17b, Remark 5.29]). For a fixed finite level structure $K \subset \mathbf{G}(\mathbb{A}_f)$, The action of $T_g$ on $\mathrm{Sh}(\mathbf{G},\mathcal{X})$ descends to a correspondence $\mathcal{T}_g \subset \mathrm{Sh}_K(\mathbf{G},\mathcal{X}) \times_E \mathrm{Sh}_K(\mathbf{G},\mathcal{X})$ given by the diagram:

$$\begin{tikzcd} \mathrm{Sh}(\mathbf{G},\mathcal{X}) \arrow[r, "T_g"] \arrow[d, "\pi_{\mathbf{G},K}"] & \mathrm{Sh}(\mathbf{G},\mathcal{X}) \arrow[d, "\pi_{\mathbf{G},K}"] \\ \mathrm{Sh}_K(\mathbf{G},\mathcal{X}) & \mathrm{Sh}_K(\mathbf{G},\mathcal{X}) \end{tikzcd}$$

The correspondence $\mathcal{T}_g = (\pi_{\mathbf{G},K})_* \cdot T_g \cdot (\pi_{\mathbf{G},K})^*$ is finite, i.e.

$$\mathcal{T}_g \in \mathcal{C}_{\mathrm{fin}}(\mathrm{Sh}_K(\mathbf{G},\mathcal{X})(\mathbb{C}) \times_E \mathrm{Sh}_K(\mathbf{G},\mathcal{X})(\mathbb{C})).$$

It is called the Hecke correspondence and is usually given by the diagram:

$$\begin{tikzcd} \mathrm{Sh}_{K_g}(\mathbf{G},\mathcal{X}) \arrow[r, "\times g"] \arrow[d, "\pi_{K_g,K}"] & \mathrm{Sh}_{K_g^{-1}}(\mathbf{G},\mathcal{X}) \arrow[d, "\pi_{K_{g^{-1}},K}"] \\ \mathrm{Sh}_K(\mathbf{G},\mathcal{X}) & \mathrm{Sh}_K(\mathbf{G},\mathcal{X}) \end{tikzcd}$$

where, we have used the notation $K_g = K \cap gKg^{-1}$ for (any) $g \in \mathbf{G}(\mathbb{A}_f)$.

# VI.9 Deligne's Reciprocity law for tori

## VI.9.1 Artin map

Let $L$ be any number field and set $\mathbf{X}_L := \mathrm{Res}_{L/\mathbb{Q}}\,\mathbb{G}_{m,L}$. Class field theory states the existence of the so called Artin map, that is a continuous surjective homomorphism

$$\mathrm{Art}_L\colon \mathbf{X}_L(\mathbb{A}_\mathbb{Q}) \twoheadrightarrow \mathrm{Gal}(L^{ab}/L),$$

sending uniformizers to geometric Frobenius elements. Let $\mathbf{X}_L(\mathbb{R})^+$ denotes the identity component for the real points and $\mathbf{X}_L(\mathbb{Q})^+ = \ker(\mathbf{X}_L(\mathbb{Q}) \to \pi_0(\mathbf{X}_L(\mathbb{R})))$. The kernel of $\mathrm{Art}_L$ is precisely the closure[14] of $\mathbf{X}_L(\mathbb{Q})\mathbf{X}_L(\mathbb{R})^+$, see [NSW08, §2 Chap. VIII]. The cited theorem affirms that the closure of the above product in the ideal class group is the kernel. Indeed the quotient map $\mathbf{X}_L(\mathbb{A}_\mathbb{Q}) \twoheadrightarrow \mathbf{X}_L(\mathbb{A}_\mathbb{Q})/\mathbf{X}_L(\mathbb{Q})$ is open and continuous, thus the preimage of the closure of $\mathbf{X}_L(\mathbb{Q})\mathbf{X}_L(\mathbb{R})^+ \subset \mathbf{X}_L(\mathbb{A}_\mathbb{Q})/\mathbf{X}_L(\mathbb{Q})$ is the image of the closure of $\mathbf{X}_L(\mathbb{Q})\mathbf{X}_L(\mathbb{R})^+ \subset \mathbf{X}_L(\mathbb{A}_\mathbb{Q})$. Recall that by definition of $\mathrm{Art}_L$, we have $\mathbf{X}_L(\mathbb{Q})$ is in its kernel and clearly $\mathbf{X}_L(\mathbb{R})^+$ is also in the kernel, therefore since the homomorphism $\mathrm{Art}_L$ is continuous and the target group $\mathrm{Gal}(L^{ab}/L)$ is Hausdorff the kernel must be closed and

[14]Here, the subgroups $\mathbf{X}_L(\mathbb{R})^+$ is identified with the subgroup of $\mathbf{X}_L(\mathbb{A}_\mathbb{Q})$ of idele elements $\mathbf{z} = (z_v)$ in $\mathbb{A}_L^\times$ such that $z_v = 1$ if the place $v$ is finite and $z_v > 0$ if the place $v$ is real.

hence contains $(\mathbf{X}_L(\mathbb{Q})\mathbf{X}_L(\mathbb{R})^+)^- \subset \mathbf{X}_L(\mathbb{A}_\mathbb{Q})$, and this shows the equality.

By the real approximation theorem (See [Mil17b, Appendix, p. 152]), $\mathbf{X}_L(\mathbb{Q})$ is dense in $\mathbf{X}_L(\mathbb{R})$ thus one has a surjective map $\mathbf{X}_L(\mathbb{Q}) \twoheadrightarrow \pi_0(\mathbf{X}_L(\mathbb{R}))$, thus $\mathbf{X}_L(\mathbb{R})/\mathbf{X}_L(\mathbb{R})^+ \simeq \pi_0(\mathbf{X}_L(\mathbb{R})) \simeq \mathbf{X}_L(\mathbb{Q})/\mathbf{X}_L(\mathbb{Q})^+ \simeq \prod_{v\in \mathrm{Hom}_\mathbb{Q}(L,\mathbb{R})}\{\pm\}$.

**Lemma** VI.9.1.1. *The closure $\mathbf{X}_L(\mathbb{Q})^- \subset \mathbf{X}_L(\mathbb{A}_f)$ is equal to $L^\times(\mathcal{O}_L^\times)^-$.*

*Proof.* Let $x = (x_v) \in \mathbf{X}_L(\mathbb{Q})^- \subset \mathbf{X}_L(\mathbb{A}_f)$ and consider $(x_n = (x_{v,n})_v)_{n\in\mathbb{N}}) \in \mathbf{X}_L(\mathbb{Q})$ a sequence that converges to $x$. There exists an open neighbourhood of $x$ of the form $\mathcal{O}_x = \mathcal{O}_x^S \times \prod_{v\notin S} \mathcal{O}_{L_v}^\times$ for some open subgroup $\mathcal{O}_x^S \subset L^{S,\times}$ such that for $n \geq N = N(\mathcal{O}_x)$ one has $x_n \in \mathcal{O}_x$ and $\ell_x := \prod_{v\in S} x_v$ lies in $\mathbf{X}_L(\mathbb{Q}) = L^\times$. For each $v \in S$, $x_{v,n}$ being convergent to $x_v$, then $|x_{v,n}|_v = |x_v|_v$ is constant for $n$ greater than some integer $N_v$ that we choose to be greater than $N$. Put for $n \geq N_x := \max_{v\in S} N_v$,

$$y_n := \ell_x^{-1} x_{v,n} = (\frac{1}{x_v} x_{v,n})_{v\in S} \otimes (x_{v,n})_{v\notin S} \in \widehat{\mathcal{O}}_L^\times \cap L^\times = \mathcal{O}_L^\times.$$

Then, one has

$$x = \ell_x \lim_{n\to\infty} y_n \in L^\times(\mathcal{O}_L^\times)^-. \quad \square$$

Consider the homomorphism

$$\mathrm{Art}_{L,f}\colon \mathbf{X}_L(\mathbb{A}_f) \to \mathbf{X}_L(\mathbb{A}_\mathbb{Q})/(\mathbf{X}_L(\mathbb{Q})\mathbf{X}_L(\mathbb{R})^+)^-, z \mapsto [(1_\infty, z)].$$

Let us show that it is surjective. Using the density of $\mathbf{X}_L(\mathbb{Q})$ in $\mathbf{X}_L(\mathbb{R})$, observe that:

$$\mathbf{X}_L(\mathbb{Q})\mathbf{X}_L(\mathbb{R})^+ = \{(x, \Delta y) \in \mathbf{X}_L(\mathbb{R}) \times \mathbf{X}_L(\mathbb{A}_f)\colon x \in \mathbf{X}_L(\mathbb{R}), y \in \mathbf{X}_L(\mathbb{Q}) \text{ s.t.}$$
$$\pi_0(x) = \pi_0(y) \in \pi_0(\mathbf{X}_L(\mathbb{R}))\}.(\star)$$

Let $z = (z_\infty, z_f) \in \mathbf{X}_L(\mathbb{A}_\mathbb{Q})$, we may modify $z$ by an element in $\mathbf{X}_L(\mathbb{Q})\mathbf{X}_L(\mathbb{R})^+$ to kill $z_\infty$. For this, consider $z' = (z_\infty^{-1}, \Delta z'_f)$ where $z'_f \in \mathbf{X}_L(\mathbb{Q})$ is any element such that $\pi_0(z'_f) = \pi_0(z'^{-1}_\infty)$ (such element exists by density of the $\mathbb{Q}$-points in the $\mathbb{R}$-points of $\mathbf{X}_L$). Hence,

$$z \mod \mathbf{X}_L(\mathbb{Q})\mathbf{X}_L(\mathbb{R})^+ = zz' \mod \mathbf{X}_L(\mathbb{Q})\mathbf{X}_L(\mathbb{R})^+,$$

and $zz' \in \mathbf{X}_L(\mathbb{A}_f)$. Accordingly

$$[z] \in \mathrm{Im}(\mathrm{Art}_{L,f}),$$

which shows the surjectivity of $\mathrm{Art}_{L,f}$. Let us show that the kernel

$$\ker \mathrm{Art}_{L,f} = \mathbf{X}_L(\mathbb{A}_f) \cap (\mathbf{X}_L(\mathbb{Q})\mathbf{X}_L(\mathbb{R})^+)^- = (\mathbf{X}_L(\mathbb{Q})^+)^-,$$

where, the super-script $^-$ denotes the closure in $\mathbf{X}_L(\mathbb{A}_\mathbb{Q})$ first, then in $\mathbf{X}_L(\mathbb{A}_f)$. Let $z \in (\mathbf{X}_L(\mathbb{A}_f) \cap \mathbf{X}_L(\mathbb{Q})\mathbf{X}_L(\mathbb{R})^+)^-$, one then has (using $\star$) a sequence of $(x_n, \Delta y_n) \in$

$\mathbf{X}_L(\mathbb{Q})\mathbf{X}_L(\mathbb{R})^+$ converging to $z$, so in particular $\lim_{n\to\infty} x_n = 1$. Hence, there is an integer $N$ big enough such that for $n \geq N$ one has $\pi_0(x_n) = \mathbf{X}_L(\mathbb{R})^+$, and accordingly $\pi_0(y_n) = \mathbf{X}_L(\mathbb{R})^+$, i.e. $y_n \in \mathbf{X}_L(\mathbb{Q})^+$. Therefore,

$$z = \lim_{n\to\infty} (x_n, y_n) = \lim_{n\to\infty} (1_\infty, y_n)$$

which shows that $z \in (\mathbf{X}_L(\mathbb{Q})^+)^-$ where the closure is taken in $\mathbf{X}_L(\mathbb{A}_f)$. This shows that $\ker \mathrm{Art}_{L,f} \subset (\mathbf{X}_L(\mathbb{Q})^+)^-$. The other inclusion is obvious. In summary, this shows that

$$\mathrm{Gal}(L^{ab}/L) \simeq \pi_0(\mathbf{X}_L(\mathbb{A}_\mathbb{Q})/\mathbf{X}_L(\mathbb{Q})) \simeq \mathbf{X}_L(\mathbb{A}_f)/(\mathbf{X}_L(\mathbb{Q})^+)^-.$$

By the above discussion we get an exact sequence

$$1 \longrightarrow (\mathbf{X}_L(\mathbb{Q})^+)^- \longrightarrow \mathbf{X}_L(\mathbb{A}_f) \xrightarrow{r_L} \mathrm{Gal}(L^{ab}/L) \longrightarrow 1. \tag{VI.3}$$

It follows from the description of the connected component of Idèle class group in [AT90, §9, Theorem 3] that the kernel of $r_L$ is isomorphic, as $\mathrm{Aut}(L/\mathbb{Q})$-module, to

$$L^{\times,+} \otimes (\mathbb{A}_f/\mathbb{Q}) \simeq L^{\times,+}(\mathcal{O}_L^{\times,+} \otimes (\mathbb{A}_f/\mathbb{Q})) \simeq L^{\times,+}(\mathcal{O}_L^{\times,+} \otimes (\widehat{\mathbb{Z}}/\mathbb{Z})).$$

yielding the exact sequence

$$1 \longrightarrow \mathcal{O}_L^{\times,+} \otimes (\mathbb{A}_f/\mathbb{Q}) \longrightarrow \mathbb{A}_{L,f}/L^{\times,+} \xrightarrow{r_L} \mathrm{Gal}(L^{ab}/L) \longrightarrow 1.$$

## VI.9.2 The cases T and Z

Now let us consider our two fields $E$ and $F$:

$$\mathrm{Art}_E\colon \mathbf{T}(\mathbb{A}_\mathbb{Q}) \twoheadrightarrow \mathrm{Gal}(E^{ab}/E), \qquad \mathrm{Art}_F\colon \mathbf{Z}(\mathbb{A}_\mathbb{Q}) \twoheadrightarrow \mathrm{Gal}(F^{ab}/F).$$

The kernel of $\mathrm{Art}_E$ (resp. $\mathrm{Art}_F$) is the closure of $\mathbf{T}(\mathbb{R})^+\mathbf{T}(\mathbb{Q}) = \mathbf{T}(\mathbb{R})\mathbf{T}(\mathbb{Q}) \simeq E^\times(\mathbb{C}^\times)^d$ (resp. $\mathbf{Z}(\mathbb{R})^+\mathbf{Z}(\mathbb{Q}) \simeq (\mathbb{R}_{>0}^\times)^d F^\times \simeq (\mathbb{R}^\times)^d(F^\times)^+)$, in $\mathbf{T}(\mathbb{A}_\mathbb{Q})^+$ (resp. $\mathbf{Z}(\mathbb{A}_\mathbb{Q})$), here we have used the fact that $E$ is totally imaginary and that $F$ is totally real. Therefore,

$$\begin{aligned}
\mathrm{Gal}(E^{ab}/E) &\simeq \pi_0(\mathbf{T}(\mathbb{A}_\mathbb{Q})/\mathbf{T}(\mathbb{Q})) \\
&\simeq \mathbf{T}(\mathbb{A}_\mathbb{Q})/\ker \mathrm{Art}_E \\
&\simeq \mathbb{A}_E^\times/(E^\times \prod_{\tilde{\iota}\in\widetilde{\Sigma}_E} \mathbb{C}_{\tilde{\iota}}^\times)^- \\
&\simeq \mathbb{A}_E^\times/E^\times(\mathcal{O}_E^\times \prod_{\tilde{\iota}\in\widetilde{\Sigma}_E} \mathbb{C}_{\tilde{\iota}}^\times)^- \\
&\simeq \mathbb{A}_E^\times/(\mathbb{C}^\times)^d\, E^\times(\mathcal{O}_E^\times)^- \\
&\simeq \mathbb{A}_{E,f}^\times/E^\times(\mathcal{O}_E^\times)^-
\end{aligned}$$

where, $(\mathcal{O}_E^\times)^-$ denotes the closure of $\mathcal{O}_E^\times$ in $\widehat{\mathcal{O}_E}^\times$. Likewise,

$$\begin{aligned}
\mathrm{Gal}(F^{ab}/F) &\simeq \pi_0(\mathbf{Z}(\mathbb{A}_\mathbb{Q})/\,\mathbf{Z}(\mathbb{Q}))\\
&\simeq \mathbf{Z}(\mathbb{A}_\mathbb{Q})/\ker \mathrm{Art}_F\\
&\simeq \mathbb{A}_F^\times/(F^\times \prod_{\iota\in\Sigma_F} \mathbb{R}_\iota)^-\\
&\simeq \mathbb{A}_F^\times/F^\times(\mathcal{O}_F^\times \prod_{\iota\in\Sigma_F} \mathbb{R}_\iota)^-\\
&\simeq \mathbb{A}_F^\times/(\mathbb{R}_{>0}^\times)^d\, F^\times(\mathcal{O}_F^\times)^-\\
&\simeq \{\pm\}^d \times \mathbb{A}_{F,f}^\times/F^\times(\mathcal{O}_F^\times)^-\\
&\simeq \mathbb{A}_{F,f}^\times/F^{\times,+}(\mathcal{O}_F^{\times,+})^-.
\end{aligned}$$

We refer the reader to [Gra14, 4.2.8 (i)] for the equalities

$$(F^\times \prod_{\iota\in\Sigma_F} F_\iota)^- = F^\times(\mathcal{O}_F^\times \prod_{\iota\in\Sigma_F} F_\iota)^- \subset \mathbb{A}_F^\times \text{ and } (E^\times \prod_{\tilde\iota\in\tilde\Sigma_E} E_{\tilde\iota}^\times)^- = E^\times(\mathcal{O}_E^\times \prod_{\tilde\iota\in\tilde\Sigma_E} E_{\tilde\iota}^\times)^- \subset \mathbb{A}_E^\times.$$

#### VI.9.2.1 Deligne's reciprocity law for $\mathbf{T}^1$

Consider the "zero"-dimensional Shimura datum $(\mathbf{T}^1, \{\det \mu_{\mathfrak{B}_\star}\})$. Using the reflex norm map computed in §VI.6

$$\nu\colon \mathbf{T} \xrightarrow{z\longmapsto \frac{z}{\bar z}} \mathbf{T}^1$$

we obtain the following reciprocity map

$$r = r(\mathbf{T}^1, \{\det \mu_{\mathfrak{B}_\star}\})\colon \mathrm{Gal}(E^{ab}/E) \longrightarrow \pi_0(\mathbf{T}^1(\mathbb{A}_\mathbb{Q})/\mathbf{T}^1(\mathbb{Q})) \simeq \mathbf{T}^1(\mathbb{A}_f)/\mathbf{T}^1(\mathbb{Q})^-,$$

by composing the inverse of $\mathrm{Art}_E\colon E^\times(\mathcal{O}_E^\times)^-\backslash\mathbb{A}_E^\times \xrightarrow{\simeq} \mathrm{Gal}(E^{ab}/E)$ and $\nu$. But, since $\mathbf{T}^1(\mathbb{Q})$ is discrete and hence closed in $\mathbf{T}^1(\mathbb{A}_f)$(See remark VI.7.0.1), the target of the map $r_{\mathrm{fin}}(\mathbf{T}^1, \{\det \mu_{\mathfrak{B}_W}\})$ is actually $\mathbf{T}^1(\mathbb{A}_f)/\mathbf{T}^1(\mathbb{Q})$.

Let $K_{\mathbf{T}^1} \subset \mathbf{T}^1(\mathbb{A}_f)$ be a open compact subgroup. The action of $\sigma \in \mathrm{Gal}(E^{ab}/E)$ on

$$[\det \mu_{\mathfrak{B}_\star}, t] \in \mathrm{Sh}_{K_{\mathbf{T}^1}}(\mathbf{T}^1, \det \mu_{\mathfrak{B}_\star})(\mathbb{C}) = \mathbf{T}^1(\mathbb{Q})\backslash(\{\det \mu_{\mathfrak{B}_\star}\} \times \mathbf{T}^1(\mathbb{A}_f)/K_{\mathbf{T}^1})$$

is defined by

$$\sigma([\det \mu_{\mathfrak{B}_\star}, t]) := [r_\infty(\sigma) \det \mu_{\mathfrak{B}_\star}, r_{\mathrm{fin}}(\sigma)\, t].$$

Here, we have $r_\infty(\sigma) \det \mu_{\mathfrak{B}_\star} = \det \mu_{\mathfrak{B}_\star}$ and the map $r$ factors then through its finite part $r_{\mathrm{fin}} = r_{\mathrm{fin}}(\mathbf{T}^1, \{\det \mu_{\mathfrak{B}_\star}\})\colon \mathbb{A}_E^\times \longrightarrow \mathbf{T}^1(\mathbb{A}_f)$.

## VI.10 Galois action on connected components

In this section we describe the action of $\mathrm{Gal}(E^{ab}/E)$ on the connected component of $\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H},\mathcal{Y})$, for a fixed open compact subgroup $K_{\mathbf{H}}$ of $\mathbf{H}(\mathbb{A}_f)$. The derived group $\mathbf{G}^{\mathrm{der}}$ is simply connected[15]. Therefore, one has a simplified description of the connected components of the Shimura variety:

$$\begin{aligned}
\pi_0(\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H},\mathcal{Y})) &= \pi_0\left(\mathbf{H}(\mathbb{Q})\backslash(\mathcal{Y}\times(\mathbf{H}(\mathbb{A}_f)/K_{\mathbf{H}}))\right)\\
&\simeq \mathbf{H}(\mathbb{Q})\backslash\mathbf{H}(\mathbb{A}_f)/K_{\mathbf{H}}\\
&= \mathbf{H}(\mathbb{Q})\backslash\mathbf{H}(\mathbb{A}_f)/K_{\mathbf{H}}\,\mathbf{H}^{\mathrm{der}}(\mathbb{A}_f)\\
&\simeq \mathbf{T}^1(\mathbb{Q})\backslash\mathbf{T}^1(\mathbb{A}_f)/\det(K_{\mathbf{H}})\\
&= \mathrm{Sh}_{\det(K_{\mathbf{H}})}(\mathbf{T}^1,\det\mu_{\mathfrak{B}_W})(\mathbb{C})
\end{aligned}$$

The previous equalities uses the connectedness of $\mathcal{Y}$ first, then density of $\mathbf{H}^{\mathrm{der}}(\mathbb{Q})$ in $\mathbf{H}^{\mathrm{der}}(\mathbb{A}_f)$ (See [Mil17b, §5 - The structure of a Shimura variety]).

Using the action of Galois group $\mathrm{Gal}(E^{ab}/E)$ on the complex points of $\mathrm{Sh}_{det(K_{\mathbf{H}})}(\mathbf{T}^1,\det\mu_{\mathfrak{B}_W})$ as described in §VI.9, we obtain that the action of $\sigma\in\mathrm{Gal}(E^{ab}/E)$ on,

$$[\det\mu_{\mathfrak{B}_W},t]\in\pi_0(\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H},\mathcal{Y}))\simeq\mathrm{Sh}_{\det(K_{\mathbf{H}})}(\mathbf{T}^1,\{\det\mu_{\mathfrak{B}_W}\})(\mathbb{C}),$$

is given by

$$\sigma\left([\det\mu_{\mathfrak{B}_W},t]\right)=[\det\mu_{\mathfrak{B}_W},r_{\mathrm{fin}}(\sigma)\,t].$$

## VI.11 Morphisms of Shimura varieties

The inclusion homomorphism $\mathbf{H}\hookrightarrow\mathbf{G}$, induces a map $\mathbf{H}(\mathbb{R})\hookrightarrow\mathbf{G}(\mathbb{R})$ that sends the Hermitian symmetric domain $\mathcal{Y}$ into $\mathcal{X}$. Therefore, we get a morphism of Shimura data $\varphi\colon(\mathbf{H},\mathcal{Y})\to(\mathbf{G},\mathcal{X})$ in the sense of [Mil17b, Definition 5.15].

**Theorem** VI.11.0.1. *The injective morphism of Shimura data* $\varphi\colon(\mathbf{H},\mathcal{Y})\to(\mathbf{G},\mathcal{X})$ *induces a closed immersion of Shimura varieties* $\mathrm{Sh}(\varphi)\colon\ \mathrm{Sh}(\mathbf{H},\mathcal{Y})\to\mathrm{Sh}(\mathbf{G},\mathcal{X})$.

*Proof.* See [Del71, Theorem 1.15]. □

[15] Indeed, $\mathbf{G}^{\mathrm{der}}_{\overline{\mathbb{Q}}}$ is isomorphic to $\mathbf{SL}_{n,\overline{\mathbb{Q}}}$.

**Remark** VI.11.0.1. *In the above theorem, the morphism* $\mathrm{Sh}(\varphi)$ *being a closed immersion of an inverse limit of regular maps means: For every "sufficiently" small compact open subgroup* $K_{\mathbf{G}} \subset \mathbf{G}(\mathbb{A}_f)$*, there is a compact open subgroup* $K_{\mathbf{H}} \subset \mathbf{H}(\mathbb{A}_f)$ *such that the map*

$$\mathrm{Sh}(\varphi)_{K_{\mathbf{H}},K_{\mathbf{G}}} \colon \mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H},\mathcal{Y}) \to \mathrm{Sh}_{K_{\mathbf{G}}}(\mathbf{G},\mathcal{X}),$$

*given naturally on* $\mathbb{C}$*-points by* $\mathbf{H}(\mathbb{Q})(y, hK_{\mathbf{H}}) \mapsto \mathbf{G}(\mathbb{Q})(y, hK_{\mathbf{G}})$ *for any* $y \in \mathcal{Y}$ *and* $h \in \mathbf{H}(\mathbb{A}_f)$*, is a closed immersion. We then get the following commutative diagram*

$$\begin{tikzcd}
\mathrm{Sh}(\mathbf{H},\mathcal{Y}) \arrow[r, "\mathrm{Sh}(\varphi)"] \arrow[d, "\pi_{\mathbf{H},K_{\mathbf{H}}}"] & \mathrm{Sh}(\mathbf{G},\mathcal{X}) \arrow[d, "\pi_{\mathbf{G},K_{\mathbf{G}}}"] \\
\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H},\mathcal{Y}) \arrow[r, "\mathrm{Sh}(\varphi)_{K_{\mathbf{H}},K_{\mathbf{G}}}"] & \mathrm{Sh}_{K_{\mathbf{G}}}(\mathbf{G},\mathcal{X}).
\end{tikzcd}$$

*To ease the notation we will omit the subscripts referring to the compacts in the map* $\mathrm{Sh}(\varphi)_{K_{\mathbf{H}},K_{\mathbf{G}}}$.

We collect here a two results concerning the algebraicity of morphisms we will be working with.

(i) The Hecke action of $\mathbf{G}(\mathbb{A}_f)$ on $\mathrm{Sh}(\mathbf{G},\mathcal{X})$ is defined over $E$ (See [Mil17b, Theorem 13.6]).

(ii) The closed immersion of Shimura varieties $\mathrm{Sh}(\varphi) \colon \mathrm{Sh}(\mathbf{H},\mathcal{Y}) \to \mathrm{Sh}(\mathbf{G},\mathcal{X})$ is defined over $E$. [Mil17b, remark 13.8].

## VI.12 The set of special cycles $\mathcal{Z}_{\mathbf{G},K}(\mathbf{H})$

For the remaining of this section, we will fix a neat compact open subgroup of $K \subset \mathbf{G}(\mathbb{A}_f)$.

**Definition** VI.12.0.1 (Special cycles). *We call a closed subvariety* $\mathcal{Z} \subset \mathrm{Sh}_K(\mathbf{G},\mathcal{X})$ *a* $\mathbf{H}$*-special cycle, if there exists an element* $g \in \mathbf{G}(\mathbb{A}_f)$ *such that* $\mathcal{Z}$ *is an irreducible component of the image of the map*

$$\mathrm{Sh}(\mathbf{H},\mathcal{Y}) \xrightarrow{\mathrm{Sh}(\varphi)} \mathrm{Sh}(\mathbf{G},\mathcal{X}) \xrightarrow{T_g} \mathrm{Sh}(\mathbf{G},\mathcal{X}) \xrightarrow{\pi_{\mathbf{G},K}} \mathrm{Sh}_K(\mathbf{G},\mathcal{X}).$$

**Lemma** VI.12.0.1. *A closed subvariety* $\mathcal{Z} \subset \mathrm{Sh}_K(\mathbf{G},\mathcal{X})$ *is a* $\mathbf{H}$*-Special cycle if and only if there exists an element* $g \in \mathbf{G}(\mathbb{A}_f)$*, such that*

$$\mathcal{Z} = [\mathcal{Y} \times gK] \subset \mathrm{Sh}_K(\mathbf{G},\mathcal{X})(\mathbb{C}).$$

*Proof.* The equivalence above follows immediately, using Remark VI.11.0.1 and by observing that

$$\mathcal{Z} = T_g \circ \mathrm{Sh}(\varphi)(\mathbf{H}(\mathbb{Q})(K_{\mathbf{H},g} \times \mathcal{Y})),$$

where $K_{\mathbf{H},g} := \mathbf{H}(\mathbb{A}_f) \cap gKg^{-1}$, See [Moo98a, Remark 2.6]. □

For every $g \in \mathbf{G}(\mathbb{A}_f)$, we will denote by $\mathfrak{z}_g$ the $n$-codimensional $\mathbf{H}$-special cycle $[\mathcal{Y} \times gK] \subset \mathrm{Sh}_K(\mathbf{G}, \mathcal{X})(\mathbb{C})$, as defined in Definition VI.12.0.1. Set,

$$\mathcal{Z}_{\mathbf{G},K}(\mathbf{H}) := \{\mathfrak{z}_g : g \in \mathbf{G}(\mathbb{A}_f).\}$$

**Lemma** VI.12.0.2. *The natural projection $\mathbf{G}(\mathbb{A}_f) \to \mathcal{Z}_{\mathbf{G},K}(\mathbf{H})$, induces the bijection*

$$\mathcal{Z}_{\mathbf{G},K}(\mathbf{H}) \simeq \mathbf{H}(\mathbb{Q})Z_{\mathbf{G}}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}_f)/K,$$

*where, $Z_{\mathbf{G}} \simeq \mathbf{T}^1 \times \mathbf{T}^1$ denotes the center of $\mathbf{G}$.*

*Proof.* Using the total geodesicity of $\mathcal{Y}$ and the Baire's category theorem, one proves that

$$\mathcal{Z}_K \simeq \mathrm{Stab}_{\mathbf{G}(\mathbb{Q})}(\mathcal{Y})\backslash\mathbf{G}(\mathbb{A}_f)/K.$$

Then, one shows that $\mathrm{Stab}_{\mathbf{G}(\mathbb{Q})}(\mathcal{Y}) = \mathbf{H}(\mathbb{Q})Z_{\mathbf{G}}(\mathbb{Q}) = N_{\mathbf{G}}(\mathbf{H})(\mathbb{Q})$. For more details see [Jet16, Lemma 2.3]. □

**Lemma** VI.12.0.3. *In the following bijection*

$$\mathcal{Z}_{\mathbf{G},K}(\mathbf{H}) \simeq \mathbf{H}(\mathbb{Q})\mathbf{Z}_{\mathbf{G}}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}_f)/K,$$

*we can replace $\mathbf{H}(\mathbb{Q})\mathbf{Z}_{\mathbf{G}}(\mathbf{H})(\mathbb{Q})$ by its closure in $\mathbf{G}(\mathbb{A}_f)$:*

$$\mathcal{Z}_{\mathbf{G},K}(\mathbf{H}) = (\mathbf{H}(\mathbb{Q})\mathbf{Z}_{\mathbf{G}}(\mathbb{Q}))^{-}\backslash\mathbf{G}(\mathbb{A}_f)/K = \mathbf{Z}_{\mathbf{G}}(\mathbb{Q})\mathbf{H}(\mathbb{Q})\mathbf{H}^{der}(\mathbb{A}_f)\backslash\mathbf{G}(\mathbb{A}_f)/K.$$

*Proof.* The second[16] equality is due to the fact that the closure of $\mathbf{H}(\mathbb{Q})$ is $\mathbf{H}(\mathbb{Q})\mathbf{H}^{\mathrm{der}}(\mathbb{A}_f)$. The latter fact is [Del79, corollaire 2.0.9], because $\mathbf{H}^{\mathrm{der}}$ verifies the strong approximation for $\{\infty\}$ since by assumption $\mathbf{H}^{\mathrm{der}}$ is simply connected, semisimple by definition and of noncompact type. Observe also, that $\mathbf{Z}_{\mathbf{G}}(\mathbb{Q})^{-} = \mathbf{Z}_{\mathbf{G}}(\mathbb{Q})$, since it is a copy of $\mathbf{T}^1(\mathbb{Q})$, which is discrete (Remark VI.7.0.1) and thus closed in $\mathbf{T}^1(\mathbb{A}_f)$. □

[16] The first equality uses the fact that if $G$ is a topological group (might even be locally compact) and $K$ an open compact subgroup (being open implies the discreteness of the quotient $G/K$), then for every subgroup $H$ of $G$ we have $H\backslash G/K = \overline{H}\backslash G/K$, where $\overline{H}$ is the closure of $H$ in $G$.

## VI.13 Fields of definition of cycles in $\mathcal{Z}_{\mathbf{G},K}(\mathbf{H})$

**Lemma** VI.13.0.1. *For every $g \in \mathbf{G}(\mathbb{A}_f)$, the cycle $\mathfrak{z}_g$ is the image of the connected component*

$$\mathbf{H}(\mathbb{Q}) \cap K_{g,\mathbf{H}} \backslash \mathcal{Y} \in \pi_0(\mathrm{Sh}_{K_{g,\mathbf{H}}}(\mathbf{H}, \mathcal{Y})).$$

*Proof.* For obvious topological reasons, $K_{g,\mathbf{H}}$ is a compact open subgroup of $\mathbf{H}(\mathbb{A}_f)$. The pre-image of $\mathfrak{z}_g$ is the component over $[(1,1)]$:

$$\mathbf{H}(\mathbb{Q}) \cap K_{g,\mathbf{H}} \backslash \mathcal{Y} \simeq \mathrm{Sh}^{\circ}_{K^{\mathrm{der}}_{\mathbf{H},g}}(\mathbf{H}^{\mathrm{der}}, \mathcal{Y}).$$

Here, $K^{\mathrm{der}}_{\mathbf{H},g} \subset \mathbf{H}^{\mathrm{der}}(\mathbb{A}_f)$ is some open compact subgroup containing $K_{\mathbf{H},g} \cap \mathbf{H}^{\mathrm{der}}(\mathbb{A}_f)$. □

Recall that, for any neat compact subgroup $K_{\mathbf{H}} \subset \mathbf{H}(\mathbb{A}_f)$, we have

$$\pi_0(\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, \mathcal{Y})) \simeq \mathbf{T}^1(\mathbb{Q}) \backslash \mathbf{T}^1(\mathbb{A}_f) / \det(K_{\mathbf{H}}).$$

The set of classes $\mathbf{T}^1(\mathbb{Q}) \backslash \mathbf{T}_{\mathbf{H}}(\mathbb{A}_f)/K_{\mathbf{T}^1}$ of $\mathbf{T}^1$ with respect to any compact open subgroup $K_{\mathbf{T}^1} \subset \mathbf{T}^1(\mathbb{A}_f)$ form an *Abelian group*. Therefore, the connected components of $\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, \mathcal{Y})$ are all defined over abelian extensions of $E$. More precisely the field of definition $E_g$ of the component $\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, \mathcal{Y})^+$ over $[(1,1)] \in \mathrm{Sh}_{\det(K_{\mathbf{H}})}(\mathbf{T}^1, \{\det \mu_{\mathfrak{B}_W}\})$ is the finite abelian extension of $E$ fixed by

$$\mathrm{Art}_E(r^{-1}(\mathbf{T}^1(\mathbb{Q})\mathbf{T}^1(\mathbb{R}) \det(K_{\mathbf{H}}))),$$

where $r = r(\mathbf{H}, \mathcal{Y}) \colon \mathbb{A}_E \longrightarrow \mathbf{T}^1(\mathbb{A}_{\mathbb{Q}})$ is the reciprocity map constructed in §VI.9 and §VI.10.

But since, for every $g \in \mathbf{G}(\mathbb{A}_f)$, the induced morphism $\mathrm{Sh}_{K_{\mathbf{H},g}}(\mathbf{H}, \mathcal{Y}) \to \mathrm{Sh}_K(\mathbf{G}, \mathcal{X})$ is defined over $E$ and every cycle $\mathfrak{z}_g$ is then defined over the subfield $E_g$ of $E^{ab}$ that satisfies (the map $r$ factors through its finite part):

$$\mathrm{Gal}(E_g/E) = \mathbf{T}^1(\mathbb{A}_f)/\mathbf{T}^1(\mathbb{Q}) \det(K_{\mathbf{H},g}).$$

# VI.14 Transfer fields and reciprocity law

In this section, we will describe a bit further the field of definition of cycles $\mathcal{Z}_{\mathbf{G},K}(\mathbf{H})$, by computing the kernel of the reciprocity map constructed in §VI.9 a map

$$r_{\mathrm{fin}}\colon \mathrm{Gal}(E^{ab}/E) \to \mathbf{T}^1(\mathbb{A}_f)/\mathbf{T}^1(\mathbb{Q})^-.$$

## VI.14.1 The kernel of the Verlagerung map

The inclusion $F \hookrightarrow E$ induces a commutative diagram

$$\begin{tikzcd} \mathbf{Z}(\mathbb{A}_f) \arrow[r, hook, "i_{F/E}"] \arrow[d, two heads, "\mathrm{Art}_F"] & \mathbf{T}(\mathbb{A}_f) \arrow[d, two heads, "\mathrm{Art}_E"] \\ \mathrm{Gal}(F^{ab}/F) \arrow[r, "\mathrm{Ver}"] & \mathrm{Gal}(E^{ab}/E), \end{tikzcd} \tag{VI.4}$$

where, Ver is the group-theoretic transfer map also called the Verlagerung map, see [Neu86, p. 26].

**Lemma** VI.14.1.1. *The kernel of the Verlagerung map is*

$$\ker(\mathit{Ver}\colon \mathit{Gal}(F^{ab}/F) \to \mathit{Gal}(E^{ab}/E)) \simeq F^\times/F^{\times,+} \simeq (\mathbb{Z}/2\mathbb{Z})^d.$$

*Proof.* We have

$$\begin{aligned} \ker(\mathrm{Art}_E) \cap \mathrm{im}(i_{E/F}) &= (\mathcal{O}_E^\times)^- E^\times \cap i_{E/F}(\mathbb{A}_{F,f}^\times) \\ &= (i_{E/F}(\mathcal{O}_F^{\times,+}))^- E^\times \cap i_{E/F}(\mathbb{A}_{F,f}^\times) \\ &\overset{(1)}{=} i_{E/F}\left((\mathcal{O}_F^{\times,+})^- F^\times \cap \mathbb{A}_{F,f}^\times\right) \\ &= i_{E/F}\left((\mathcal{O}_F^{\times,+})^- F^\times\right) \end{aligned}$$

where, we have used $E \cap i_{E/F}(\mathbb{A}_{F,f}^{\times,+}) = i_{E/F}(F^\times)$ for (1) and then Dirichlet's unit theorem; since $[\mathcal{O}_E^\times : \mathcal{O}_F^\times] < \infty$, let $t_1, \cdots, t_?$ be a set of coset representatives for $\mathcal{O}_E^\times/\mathcal{O}_F^\times$, then

$$E^\times(\mathcal{O}_E^\times)^- = E^\times(\cup_i t_i \mathcal{O}_F^\times)^- = E^\times(\mathcal{O}_F^\times)^- = E^\times(\mathcal{O}_F^{\times,+})^-.$$

By commutativity of the above diagram (and injectivity of $i_{E/F}$) we get

$$\begin{aligned} \ker(\mathrm{Ver}) &= \mathrm{Art}_F(i_{F/E}^{-1}(\ker(\mathrm{Art}_E) \cap \mathrm{im}(i_{E/F}))) \\ &= \mathrm{Art}_F((\mathcal{O}_F^{\times,+})^- F^\times). \end{aligned}$$

Thus,

$$\ker(\mathrm{Ver}) \simeq (\mathcal{O}_F^{\times,+})^- F^\times / \ker(\mathrm{Art}_F) = (\mathcal{O}_F^{\times,+})^- F^\times / (\mathcal{O}_F^{\times,+})^- F^{\times,+}.$$

Accordingly, (See footnote 7 page 193)

$$\ker(\mathrm{Ver}) \simeq F^\times/F^{\times,+}.$$

Finally, we get $F^\times/F^{\times,+} \simeq (\mathbb{Z}/2\mathbb{Z})^d$ via the archimedean signature map:

$$\mathrm{sgn}_\infty\colon F^\times \to F_{\mathbb{R}}^\times/(F_{\mathbb{R}}^\times)^2 \simeq \prod_{\iota\in\Sigma_F}\{\pm 1\}, \quad z \mapsto (\iota(z)/|\iota(z_\iota)|)_{\iota\in\Sigma_F},$$

which is surjective with kernel $F^{\times,+}$. □

## VI.14.2 The kernel of the reciprocity map

**Proposition** VI.14.2.1. *We have a long exact sequence*

$$F^\times/F^{\times,+} \hookrightarrow Gal(F^{ab}/F) \xrightarrow{Ver} Gal(E^{ab}/E) \xrightarrow{r_{\mathrm{fin}}} \mathbf{T}^1(\mathbb{A}_f)/\mathbf{T}^1(\mathbb{Q}).$$

*Proof.* As we have already computed the kernel of the Verlagerung map in Lemma VI.14.1.1, it remains to show that the kernel of the map $r_{\mathrm{fin}}$ is the image of the Verlagerung map. Recall that we have an exact sequence (see VI.2)

$$1 \longrightarrow \mathbf{Z} \longrightarrow \mathbf{T} \xrightarrow{\nu\colon z\mapsto\frac{z}{\bar z}} \mathbf{T}^1 \longrightarrow 1. \tag{VI.5}$$

Using Galois cohomology and Hilbert Theorem 90, we deduce from it the following exact diagram

$$\begin{tikzcd} 1 \arrow[r] & \mathbf{Z}(\mathbb{Q}) \arrow[r] \arrow[d] & \mathbf{T}(\mathbb{Q}) \arrow[r, "\nu"] \arrow[d] & \mathbf{T}^1(\mathbb{Q}) \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \mathbf{Z}(\mathbb{A}_\mathbb{Q}) \arrow[r] & \mathbf{T}(\mathbb{A}_\mathbb{Q}) \arrow[r, "\nu"] & \mathbf{T}^1(\mathbb{A}_\mathbb{Q}) \arrow[r] & 1. \end{tikzcd}$$

Taking the closure in the adelic points of the first row yields the following commutative diagram[17] and using the exact sequence (VI.3) seen in §VI.9.1, we get the following commutative diagram, where the lower right square is the definition of $r_{\mathrm{fin}} = r(\mathbf{T}^1, \{\det \mu_{\mathfrak{B}_W}\})$

[17] Recall that $\mathbf{T}^1(\mathbb{Q})$ is discrete in $\mathbf{T}^1(\mathbb{A}_f)$ by Remark VI.7.0.1

as in §VI.9

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & . \\
1 \arrow[r] & (\mathbf{Z}(\mathbb{Q}))^- \arrow[r] \arrow[d] & \mathbf{T}(\mathbb{Q})^- \arrow[r, "\nu"] \arrow[d] & \mathbf{T}^1(\mathbb{Q}) \arrow[r] \arrow[d] & 1. \\
1 \arrow[r] & \mathbf{Z}(\mathbb{A}_f) \arrow[r] & \mathbf{T}(\mathbb{A}_f) \arrow[r, "\nu"] \arrow[d, "\mathrm{Art}_E"'] & \mathbf{T}^1(\mathbb{A}_f) \arrow[r] \arrow[d] & 1 \\
 & & \mathrm{Gal}(E^{ab}/E) \arrow[r, "r_{\mathrm{fin}}"] \arrow[d] & \mathbf{T}^1(\mathbb{A}_f)/\mathbf{T}^1(\mathbb{Q}) \arrow[r] \arrow[d] & 1 \\
 & & 1 & 1 &
\end{tikzcd}$$

Therefore,

$$\begin{aligned}
\ker(r_{\mathrm{fin}}) &= \mathrm{Art}_E(\nu^{-1}(\mathbf{T}^1(\mathbb{Q}))) \\
&= \mathrm{Art}_E \circ i_{E/F}(\mathbf{Z}(\mathbb{A}_f)\mathbf{T}(\mathbb{Q})^-) \\
&= \mathrm{Art}_E(i_{E/F}(\mathbf{Z}(\mathbb{A}_f))) \\
&= \mathrm{Ver} \circ \mathrm{Art}_F(\mathbf{Z}(\mathbb{A}_f)) && \text{(VI.4)} \\
&= \mathrm{Ver}(\mathrm{Gal}(F^{ab}/F)) \simeq \mathbb{A}_{F,v}^\times E^\times/E^\times \simeq \mathbb{A}_{F,f}^\times/F^\times. && \square
\end{aligned}$$

### VI.14.3 Transfer fields of definition

In the remainder of this section, we focus on the relation between $E(\infty)$ and ring class fields. For every $\mathcal{O}_F$-order $\mathcal{O}$ in $\mathcal{O}_E$, we denote by $\widehat{\mathcal{O}}^\times$ the group of units of its profinite completion. By extending $1 \in \mathcal{O}_E$ to a $\mathcal{O}_F$-basis we see that such order necessarily of the form $\mathcal{O} = \mathcal{O}_F + \mathfrak{c}\mathcal{O}_E$ where $\mathfrak{c} \subset \mathcal{O}_F$ is a non-zero ideal of $\mathcal{O}_F$ and

$$\widehat{\mathcal{O}}_{\mathfrak{c}}^\times = (\mathcal{O}_{\mathfrak{c}} \otimes \widehat{\mathbb{Z}})^\times = (\widehat{\mathcal{O}}_F + \mathfrak{c}\widehat{\mathcal{O}}_E)^\times \subset \mathbb{A}_{E,f}^\times = (E \otimes \widehat{\mathbb{Z}})^\times.$$

**Definition** VI.14.3.1. *We attach to each $\mathcal{O}_F$-order $\mathcal{O}_{\mathfrak{c}} = \mathcal{O}_F + \mathfrak{c}\mathcal{O}_E$ two subfields $E(\mathfrak{c}) \subset E[\mathfrak{c}] \subset E^{ab}$:*

1. *The* ring class field of conductor $\mathfrak{c}$ *denoted $E[\mathfrak{c}]$, is the fixed field of* $\mathrm{Art}_E(\widehat{\mathcal{O}}_{\mathfrak{c}}^\times E^\times/E^\times)$, *i.e.*
$$Gal(E[\mathfrak{c}]/E) \simeq E^\times \backslash \mathbb{A}_{E,f}^\times/\widehat{\mathcal{O}}_{\mathfrak{c}}^\times \simeq \mathrm{Pic}(\mathcal{O}_{\mathfrak{c}}).$$
2. *The* transfer field of conductor $\mathfrak{c}$ *denoted $E(\mathfrak{c})$, is the field whose norm subgroup is $E^\times \cdot \mathbb{A}_{F,f}^\times \cdot \widehat{\mathcal{O}}_{\mathfrak{c}}^\times$, i.e.*
$$Gal(E(\mathfrak{c})/E) \simeq E^\times \mathbb{A}_{F,f}^\times \backslash \mathbb{A}_{E,f}^\times/\widehat{\mathcal{O}}_{\mathfrak{c}}^\times.$$

*We have,*

$$Gal(E[\mathcal{O}_\mathfrak{c}]/E(\mathcal{O}_\mathfrak{c})) \simeq \frac{E^\times \mathbb{A}_{F,f}^\times \widehat{\mathcal{O}}_\mathfrak{c}^\times}{E^\times \widehat{\mathcal{O}}_\mathfrak{c}^\times} \simeq \frac{\mathbb{A}_{F,f}^\times}{\mathbb{A}_{F,f}^\times \cap E^\times \widehat{\mathcal{O}}_\mathfrak{c}^\times}.$$

*which is a quotient of* $\mathrm{Pic}(\mathcal{O}_F)$*.*

Now, If $F$ has class number one (e.g. $\mathbb{Q}$), then for any $\mathcal{O}_F$-order $\mathcal{O}$ we have $E[\mathcal{O}] = E(\mathcal{O})$. Moreover, (VI.7) implies that $E(\infty) = E[\infty]$. For $\mathfrak{c}, \mathfrak{c}'$ two non-zero ideals of $\mathcal{O}_F$, we have $\widehat{\mathcal{O}}_\mathfrak{c}^\times \cdot \widehat{\mathcal{O}}_{\mathfrak{c}'}^\times = \widehat{\mathcal{O}}_{\mathrm{gcd}(\mathfrak{c},\mathfrak{c}')}^\times$ and

$$E^\times \mathbb{A}_{F,f}^\times \widehat{\mathcal{O}}_{\mathrm{lcm}(\mathfrak{c},\mathfrak{c}')}^\times \subset E^\times \mathbb{A}_{F,f}^\times \widehat{\mathcal{O}}_\mathfrak{c}^\times \cap E^\times \mathbb{A}_{F,f}^\times \widehat{\mathcal{O}}_\mathfrak{c}^\times, \quad E^\times \widehat{\mathcal{O}}_{\mathrm{lcm}(\mathfrak{c},\mathfrak{c}')}^\times \subset E^\times \widehat{\mathcal{O}}_\mathfrak{c}^\times \cap E^\times \widehat{\mathcal{O}}_\mathfrak{c}^\times$$

thus,

$$E[\mathfrak{c}] \cap E[\mathfrak{c}'] = E[\mathrm{gcd}(\mathfrak{c}, \mathfrak{c}')], \quad E(\mathfrak{c}) \cap E(\mathfrak{c}') = E(\mathrm{gcd}(\mathfrak{c}, \mathfrak{c}')) \tag{VI.6}$$

and

$$E[\mathfrak{c}]E[\mathfrak{c}'] \subset E[\mathrm{lcm}(\mathfrak{c}, \mathfrak{c}')], \quad E(\mathfrak{c})E(\mathfrak{c}') \subset E(\mathrm{lcm}(\mathfrak{c}, \mathfrak{c}')).$$

Set $E[\infty] = \cup_\mathcal{O} E[\mathcal{O}]$ and $E(\infty) = \cup_\mathcal{O} E(\mathcal{O})$, where the union is taken over all $\mathcal{O}_F$-orders of $\mathcal{O}_E$. There exists a descending chain of $\mathcal{O}_F$-orders $\{\mathcal{O}_{\mathfrak{c}_i}\}_{i\geq 1}$ such that

$$\widehat{\mathcal{O}}_F^\times = \bigcap_i \widehat{\mathcal{O}}_{\mathfrak{c}_i}^\times, E[\infty] = \bigcup_i E[\widehat{\mathcal{O}}_{\mathfrak{c}_i}^\times] \text{ and } E(\infty) = \bigcup_i E(\widehat{\mathcal{O}}_{\mathfrak{c}_i}^\times).$$

Accordingly, $\mathrm{Art}_E(\widehat{\mathcal{O}}_F^\times) = \cap_i \mathrm{Art}_E(\widehat{\mathcal{O}}_{\mathfrak{c}_i}^\times)$. From which one deduces that

$$E[\infty] = (E^{ab})^{\mathrm{Art}_E(\widehat{\mathcal{O}}_F^\times)} \text{ and } E(\infty) = (E^{ab})^{\mathrm{Art}_E(\mathbb{A}_{F,f}^\times)}.$$

In other words, the transfer field $E(\infty)$ is the subfield of $E^{ab}$ fixed by $\mathrm{Ver}(\mathrm{Gal}(F^{ab}/F))$. The field $E[\infty]$ is a finite Galois extension of $E(\infty)$, with

$$\mathrm{Gal}(E[\infty]/E(\infty)) \simeq F^\times \backslash \mathbb{A}_{F,f}^\times / \widehat{\mathcal{O}}_F^\times \simeq \mathrm{Pic}(\mathcal{O}_F). \tag{VI.7}$$

Moreover, the extension $E(\infty)/F$ is Galois and its Galois group $\mathrm{Gal}(E(\infty)/F)$ is the Galois dihedral extension:

$$1 \longrightarrow \mathrm{Gal}(E(\infty)/E) \longrightarrow \mathrm{Gal}(E(\infty)/F) \longrightarrow \mathrm{Gal}(E/F) \longrightarrow 1$$

equipped with the canonical splitting given by the complex conjugation

$$\mathrm{Gal}(E(\infty)/F) \simeq \mathrm{Gal}(E(\infty)/E) \rtimes \{1, c\}.$$

From now on, we will use the notation

$$\mathrm{Art}_E^1 \colon \mathbf{T}^1(\mathbb{A}_f)/\mathbf{T}^1(\mathbb{Q}) \xrightarrow{\simeq} \mathrm{Gal}(E(\infty)/E),$$

for the inverse image of $r_{\mathrm{fin}}(\mathbf{T}^1, \{\det \mu_{\mathfrak{B}_W}\})$. Proposition VI.14.2.1 has the following immediate consequence:

**Corollary** VI.14.3.1. *For every $g \in \mathbf{G}(\mathbb{A}_f)$, the field of definition $E_g$ of the cycle $\mathfrak{z}_g \in \mathcal{Z}_{\mathbf{G},K}(\mathbf{H})$ is contained in the transfer field $E(\infty)$.*

**Remark** VI.14.3.1. *We warn the reader, that there exists two notions of "ring class fields" in the literature. Our use of this terminology follows the one used in [CV05, CV07]. The second terminology, which is used in [Zha01, Nek07], is what we called "transfer fields".*

## VI.15 Galois action via $\mathbf{H}(\mathbb{A}_f)$

**Proposition** VI.15.0.1. *For every $\sigma \in Gal(E(\infty)/E)$, let $h_\sigma \in \mathbf{H}(\mathbb{A}_f)$ be any element satisfying $\mathrm{Art}_E^1(\det(h_\sigma)\cdot \mathbf{T}^1(\mathbb{Q})) = \sigma|_{E(\infty)}$. For every $g \in \mathbf{G}(\mathbb{A}_f)$, we have*

$$\sigma(\mathfrak{z}_g) = \mathfrak{z}_{h_\sigma g}.$$

*Proof.* Fix an element $g \in \mathbf{G}(\mathbb{A}_f)$. In §VI.10, we saw the description of the action of $\mathrm{Gal}(E(\infty)/E)$ on the set of connected components of $\mathrm{Sh}_{K_{\mathbf{H},g}}(\mathbf{H},\mathcal{Y})(\mathbb{C})$:

$$\bigsqcup_{h\in\mathcal{C}_{\mathbf{H},g}} \Gamma_h\backslash\mathcal{Y},$$

where $\mathcal{C}_{\mathbf{H},g} = \mathbf{H}(\mathbb{Q})\backslash\mathbf{H}(\mathbb{A}_f)/K_{\mathbf{H},g}$ and $\Gamma_h = \mathbf{H}(\mathbb{Q}) \cap hK_{\mathbf{H},g}h^{-1} = \mathbf{H}(\mathbb{Q}) \cap K_{\mathbf{H},hg}$.

This description says that, for every $\sigma \in \mathrm{Gal}(E(\infty)/E)$, if we let $h_\sigma \in \mathbf{H}(\mathbb{A}_f)$ be any element verifying $\mathrm{Art}_E^1(\det(h_\sigma)\cdot\mathbf{T}^1(\mathbb{Q})) = \sigma|_{E(\infty)}$, then

$$(\Gamma_h\backslash\mathcal{Y})^\sigma = \Gamma_{h_\sigma h}\backslash\mathcal{Y}.$$

On the other hand by Lemma VI.13.0.1, the cycle $\mathfrak{z}_g$ is the image of $\Gamma_1\backslash\mathcal{Y} = K_{\mathbf{H},g}\backslash\mathcal{Y}$ by the closed immersion (Remark VI.11.0.1) $\mathrm{Sh}(\Phi)_{K_\mathbf{H},K_\mathbf{G}}\colon \mathrm{Sh}_{K_{\mathbf{H},g}}(\mathbf{H},\mathcal{Y}) \to \mathrm{Sh}_{K_\mathbf{G}}(\mathbf{G},\mathcal{X})$. Reccall that $\mathrm{Sh}(\Phi)_{K_\mathbf{H},K_\mathbf{G}}$ is defined over $E$. Therefore, $\sigma(\mathfrak{z}_g) = \mathfrak{z}_{h_\sigma g}$. □

Consequently, the left action of $\mathbf{H}(\mathbb{A}_f)$ on the set of $\mathbf{H}$-special cycles

$$\mathcal{Z}_{\mathbf{G},K}(\mathbf{H}) = \mathbf{Z}_\mathbf{G}(\mathbb{Q})\mathbf{H}(\mathbb{Q})\mathbf{H}^{\mathrm{der}}(\mathbb{A}_f)\backslash\mathbf{G}(\mathbb{A}_f)/K_\mathbf{G},$$

descends to an action of

$$\mathbf{H}(\mathbb{Q})\mathbf{H}^{\mathrm{der}}(\mathbb{A}_f)\backslash\mathbf{H}(\mathbb{A}_f) \simeq \mathbf{T}^1(\mathbb{Q})\backslash\mathbf{T}^1(\mathbb{A}_f) \simeq \mathrm{Gal}(E(\infty)/E),$$

which yields,

$$\mathrm{Gal}(E(\infty)/E)\backslash\mathcal{Z}_{\mathbf{G},K}(\mathbf{H}) \simeq \mathbf{Z}_\mathbf{G}(\mathbb{Q})\mathbf{H}(\mathbb{A}_f)\backslash\mathbf{G}(\mathbb{A}_f)/K.$$

Using the above proposition, we see that for every $g \in \mathbf{G}(\mathbb{A}_f)$, the cycle $\mathfrak{z}_g$ is defined over the subfield $E'_g \subset E(\infty)$, given by[18]

$$\mathrm{Gal}(E(\infty)/E'_g) = \mathrm{Art}^1_E(\det\big((Z_{\mathbf{G}}(\mathbb{Q})\mathbf{H}(\mathbb{Q})\mathbf{H}^{\mathrm{der}}(\mathbb{A}_f)gKg^{-1}) \cap \mathbf{H}(\mathbb{A}_f)\big)\,.$$

## VI.16 Hecke action on $\mathcal{Z}_{\mathbf{G},K}(\mathbf{H})$

Consider the map

$$\mathcal{T}\colon \mathbf{G}(\mathbb{A}_f) \longrightarrow \mathcal{C}_{\mathrm{fin}}(\mathrm{Sh}_K(\mathbf{G},\mathcal{X})(\mathbb{C}) \times_E \mathrm{Sh}_K(\mathbf{G},\mathcal{X})(\mathbb{C}))$$
$$g \longmapsto \mathcal{T}_g$$

where, $\mathcal{T}_g$ is the Hecke correspondence defined in §VI.8. The map $\mathcal{T}$ factors through the double quotient $K\backslash\mathbf{G}(A_f)/K$.

For any $g \in \mathbf{G}(A_f)$, we choose a system $(g_i)$ of representatives of $KgK/K$, i.e. $KgK = \sqcup_i g_i K \in \mathbf{G}(\mathbb{A}_f)/K$. We get then an operator on $\mathbb{Z}[\mathcal{Z}_{\mathbf{G},K}(\mathbf{H})]$ defined as follows

$$\mathcal{T}_g\colon \mathbb{Z}[\mathcal{Z}_{\mathbf{G},K}(\mathbf{H})] \longrightarrow \mathbb{Z}[\mathcal{Z}_{\mathbf{G},K}(\mathbf{H})], \qquad \mathfrak{z}_{g'} \longmapsto \textstyle\sum_i \mathfrak{z}_{g'g_i}.$$

Define $\mathcal{H}_K = \mathcal{H}(\mathbf{G}(\mathbb{A}_f)/\!\!/K)$ to be the global Hecke algebra generated over $\mathbb{Z}$ by $\{KgK\}$ for all $g \in \mathbf{G}(\mathbb{A}_f)$, equipped with the classical convolution product. By definition, the actions of $\mathrm{Gal}(E(\infty)/E)$ and $\mathcal{H}_K$ on $\mathbb{Z}[\mathcal{Z}_{\mathbf{G},K}(\mathbf{H})]$ commute. This yields the $\mathrm{Gal}(E(\infty)/E) \times \mathcal{H}_K$-module $\mathbb{Z}[\mathcal{Z}_{\mathbf{G},K}(\mathbf{H})]$.

[18] The field $E'_g$ is contained in the field $E_g$ defined at the end of §VI.13.

# CHAPTER VII

# DISTRIBUTION RELATIONS

# VII.1 Main theorems on distribution relations

## VII.1.1 Notations

We will systematically use notation from the previous chapter.

For any finite set $S$ of finite places of our fixed totally real field $F$, set $\mathcal{O}_F^S$ for the ring of $S$-units that is the set of $x \in F$ such that $v(x) \geq 0$ for all finite places $v \notin S$, and put $\mathcal{O}_E^S = \mathcal{O}_E \otimes_{\mathcal{O}_F} \mathcal{O}_F^S$. We will also use

$$F_S := \prod_{v \in S} F_v, \quad \text{and} \quad \mathbb{A}_{F,f}^S := \prod_{v \notin S}{}' F_v$$

the restricted product of the additive groups $F_v^\times$ for all finite places $v \notin S$, with respect to the local integers $\mathcal{O}_{F_v}^\times$. One can write the finite adeles of $F$ as the product $\mathbb{A}_{F,f} = F_S \times \mathbb{A}_{F,f}^S$. For any place $v$ of $F$, let $F_v$ be the completion of $F$ at $v$ and let $\mathcal{O}_{F_v}$ be its ring of integers with uniformizer $\varpi_v$ and maximal ideal $\mathfrak{p}_v = \varpi_v \mathcal{O}_{F_v}$. For any $F$-algebra $R$, let $R_v = R \otimes_F F_v$.

Recall that we have fixed in §VI.1, for each finite place $v$ of $F$, an embedding $\iota_v \colon \overline{F} \hookrightarrow \overline{F}_v$ , and we set $w_v$ for the unique place of $E$ defined by $\iota_v$, If $v$ splits in $E$, by abuse of notation denote the other place by $\overline{w}_v$. When the place $v$ is understood from the context, we will omit the subscript $v$ and simply write $w$ and $\overline{w}$.

## VII.1.2 Compact subgroups and base cycles

### VII.1.2.1 Integral models

Recall that we identify $\mathbf{U}(W)$ with the subgroup of $\mathbf{U}(V)$ given by

$$\{g \in \mathbf{U}(V)(R) \subset \mathbf{GL}(V \otimes_F R) \colon g \cdot x = x, \quad \forall x \in D \otimes_F R\},$$

for any $F$-algebras $R$. We have a faithful representations of $\mathbf{U}(V)$ and $\mathbf{U}(W)$

$$\mathbf{U}(W) \xhookrightarrow{\iota} \mathbf{U}(V) \hookrightarrow \mathbf{GL}(V_F)$$

where $V_F$ is the underlying $F$-vector space of the hermitian $E$-space $(V, \psi)$ (§VI.2). We identify $\mathbf{U}_V$ and $\mathbf{U}_W$ with closed subgroups of $\mathbf{GL}(V_F)$. Let $\underline{\mathbf{U}}_V$ and $\underline{\mathbf{U}}_W$ be the schematic closures of $\mathbf{U}(V)$ and $\mathbf{U}(W)$ in $\mathcal{G} = \mathbf{GL}(V_F)_{\mathcal{O}_F}$ [1].

For $\star \in \{V, W\}$, the schematic closure $\underline{\mathbf{U}}_\star$ is a model for $\mathbf{U}(\star)$ over $\mathcal{O}_F$ [GH19, Lemma 2.4.1]. But by [GH19, Lemma 2.4.2], there is a finite set $S^0_\star$ of finite places of $F$ such that $\underline{\mathbf{U}}_{\star,\mathcal{O}_F^{S^0_\star}}$ becomes smooth over $\mathcal{O}_F^{S^0_\star}$. Subsequently, [Con14, Proposition 3.1.9] ensures that for a large enough finite set $S^1_\star$ of finite places of $F$ containing $S^0_\star$, all the fibers of $\underline{\mathbf{U}}_{\star,\mathcal{O}_F^{S^1_\star}}$ will be connected and reductive, i.e. $\underline{\mathbf{U}}_{\star,\mathcal{O}_F^{S^1_\star}}$ is a reductive $\mathcal{O}_F^{S^1_\star}$-model of $\mathbf{U}(\star)$. Accordingly, the homomorphism $\iota$ extends to the models over $\mathcal{O}_{F,S^1}$ for $S^1 = S^1_V \cup S^1_W$, and we get

$$\underline{\mathbf{U}}_{W,\mathcal{O}_F^{S^1}} \xhookrightarrow{\iota} \underline{\mathbf{U}}_{V,\mathcal{O}_F^{S^1}} \hookrightarrow \mathcal{G}_{\mathcal{O}_{F,S^1}}.$$

Using [GH19, Lemma 2.4.1], we get for any finite place $v \notin S^1$ of $F$ a hyperspecial maximal compact subgroup

$$\begin{aligned} \underline{\mathbf{U}}_V(\mathcal{O}_{F_v}) &= \mathbf{U}(V)(F_v) \cap \mathcal{G}(\mathcal{O}_{F_v}) \text{ and,} \\ \underline{\mathbf{U}}_W(\mathcal{O}_{F_v}) &= \mathbf{U}(W)(F_v) \cap \mathcal{G}(\mathcal{O}_{F_v}) = \mathbf{U}(W)(F_v) \cap \underline{\mathbf{U}}_V(\mathcal{O}_{F_v}). \end{aligned} \tag{VII.1}$$

**Remark** VII.1.2.1. *In the previous discussion we chose on purpose to use general arguments to stay relatively in line with Remark VII.3.2.1. Nevertheless, a more specific argument goes as follows: We pick an $\mathcal{O}_E$ lattice $\mathcal{L}_W$ in $W$, $\mathcal{L}_D$ in $D$ and set $\mathcal{L}_V = \mathcal{L}_W \oplus \mathcal{L}_D$ in $V$. We choose them such that they are contained in their duals, with the quotients as small as possible. Now, we take $S^1$ to be the bad places, those occurring in the quotient $\mathcal{L}_V^\vee/\mathcal{L}_V$. Away from $S^1$, the restriction of $\psi$ on $\mathcal{L}_V$, $\mathcal{L}_W$ is a perfect integral hermitian pairing, giving rise to the desired smooth reductive models which are unitary groups.*

#### VII.1.2.2 Compact subgroups

Let $K_\star$, with $\star \in \{V, W\}$, be any open compact subgroup of $\underline{\mathbf{U}}_\star(\mathbb{A}_{F,f})$. It inersects then $\underline{\mathbf{U}}_\star(\prod_{v\notin S^1} \mathcal{O}_{F_v}) = \prod_{v\notin S^1} \underline{\mathbf{U}}_\star(\mathcal{O}_{F_v})$ in an open compact subgroup. Therefore, there exists a finite set $S^2 = S^2(K_V, K_W)$[2] containing $S^1$, for which both compact subgroups can be written as

$$K_\star = K_{\star,S^2} \times \prod_{v\notin S^2} \underline{\mathbf{U}}_\star(\mathcal{O}_{F_v}), \forall \star \in \{V, W\}$$

where $K_{\star,S^2}$ is some open compact subgroup of $\mathbf{U}(\star)(F_{S^2}) = \prod_{v\in S^2} \underline{\mathbf{U}}_\star(F_v)$. In particular, if we set $K_{\star,v} := \underline{\mathbf{U}}_\star(\mathcal{O}_{F_v})$ for all finite places $v$ of $F$ away from $S^2$, we have by (VII.1)

$$K_{W,v} = \mathbf{U}(W)(F_v) \cap K_{V,v}.$$

[1] Here, we are picking up implicitly a lattice in $V_F$ to obtain the $\mathcal{O}_F$ structure.
[2] We want to insist here that the enlarged set $S$ depends on both compacts $K_V$ and $K_W$.

#### VII.1.2.3 A base cycle

Now for each $\star \in \{V, W\}$, let $g_\star = (g_{\star,v})$ be any fixed element of $\mathbf{U}_\star(\mathbb{A}_f)$. By definition of the adelic points of $\mathbf{U}_\star$, there exists a finite set of places $S = S(K_V, K_W, g_V, g_W)$ containing $S^2 \cup \mathrm{Ram}(E/F)$ and such that $g_{\star,v} \in K_{\star,v}$ for all $v \notin S$ and $g_{\star,S} \in \mathbf{U}_\star(F_S)$. But as far as the formation of the cycle $\mathfrak{z}_g$ for $g = (g_V, g_W)$ is concerned, nothing is lost by considering $g_0 = (g_{0,V}, g_{0,W}) \in \mathbf{G}(\mathbb{A}_f)$, such that $g_{0,\star}^S = 1 \in \mathbf{U}_\star(\mathbb{A}_{F,f}^S)$ and $g_{0,\star,S} = g_{\star,S} \in \mathbf{U}_\star(F_S)$, since $\mathfrak{z}_g = \mathfrak{z}_{g_0}$.

Now that we have fixed our set of finite places $S$, let us consider

$$\underline{\mathbf{T}}^1 := \ker \mathrm{Norm}\colon \underbrace{\mathrm{Res}_{\mathcal{O}_E^S/\mathcal{O}_F^S} \mathbb{G}_{\mathcal{O}_E^S}}_{:=\underline{\mathbf{T}}} \to \underbrace{\mathbb{G}_{\mathcal{O}_F^S}}_{:=\underline{\mathbf{Z}}}.$$

Each of the above reductive groups over $\mathcal{O}_F^S$ are models for the obvious corresponding groups in $\mathbf{T}_F^1, \mathbf{T}_F$ and $\mathbf{Z}_F$, so in particular

$$\mathbf{T}^1(\mathbb{A}_f) = \underline{\mathbf{T}}^1(\mathbb{A}_{F,f}), \mathbf{T}(\mathbb{A}_f) = \underline{\mathbf{T}}(\mathbb{A}_{F,f}) \text{ and } \mathbf{Z}(\mathbb{A}_f) = \underline{\mathbf{Z}}(\mathbb{A}_{F,f}).$$

We view again $\underline{\mathbf{T}}^1$ as the center of $\underline{\mathbf{U}}_W$ and also $\underline{\mathbf{U}}_V$. Let $\underline{\mathbf{U}}_W^{\mathrm{der}}$ denote the kernel of the determinant map $\det\colon \underline{\mathbf{U}}_W \to \underline{\mathbf{T}}^1$. Write $\underline{\nu}\colon \underline{\mathbf{T}} \twoheadrightarrow \underline{\mathbf{T}}^1$ for the homomorphism given on $\mathcal{O}_F^S$-points by $z \mapsto z/\overline{z}$.

Define $\underline{\mathbf{G}} := \underline{\mathbf{U}}_V \times \underline{\mathbf{U}}_W$ and $\underline{\mathbf{H}} := \Delta(\underline{\mathbf{U}}_W) \subset \underline{\mathbf{G}}$[3]. We then have

$$\mathbf{G}(\mathbb{A}_f) = \underline{\mathbf{G}}(\mathbb{A}_{F,f}), \quad \text{and} \quad \mathbf{H}(\mathbb{A}_f) = \underline{\mathbf{H}}(\mathbb{A}_{F,f}).$$

Set $K_v := K_{V,v} \times K_{W,v}$ for any place $v \notin S$, $K_S = K_{V,S} \times K_{W,S}$, $K^S = K_V^S \times K_W^S$. $K = K_V \times K_W$ and $K_{\mathbf{H}} = \Delta(K_W) = K_{\mathbf{H},S} \times K_{\mathbf{H}}^S$.

**Remark** VII.1.2.2. *For each place $v \notin S$, we have an exact sequence of $\mathcal{O}_{F_v}$-groups*

$$1 \longrightarrow \underline{\mathbf{U}}_\star^{\mathrm{der}} \longrightarrow \underline{\mathbf{U}}_\star \xrightarrow{\det} \underline{\mathbf{T}}^1 \longrightarrow 1 .$$

*It induces the long exact sequence in étale cohomology*

$$1 \longrightarrow \underline{\mathbf{U}}_\star^{\mathrm{der}}(\mathcal{O}_{F_v}) \longrightarrow \underline{\mathbf{U}}_\star(\mathcal{O}_{F_v}) \xrightarrow{\det} \underline{\mathbf{T}}^1(\mathcal{O}_{F_v}) \longrightarrow H^1_{\acute{e}t}(\mathcal{O}_{F_v}, \underline{\mathbf{U}}_\star^{\mathrm{der}}).$$

*On one hand, we have $H^1_{\acute{e}t}(\mathcal{O}_{F_v}, \underline{\mathbf{U}}_\star^{\mathrm{der}}) = H^1(\mathbb{F}_{q_v}, \underline{\mathbf{U}}_\star^{\mathrm{der}})$ [Mil80, 4.5 §III]. On the other hand, Lang's theorem implies $H^1(\mathbb{F}_{q_v}, \underline{\mathbf{U}}_\star^{\mathrm{der}}) = 0$ [Lan56]. Therefore,*

$$\det(\underline{\mathbf{U}}_\star(\mathcal{O}_{F_v})) = \underline{\mathbf{T}}^1(\mathcal{O}_{F_v}). \quad \square$$

[3]The diagonal homomorphism $\Delta$ extends to the models over $\mathcal{O}_F^S$ since $\iota$ does.

### VII.1.3 The field $\mathcal{K}$

The stabilizer of $\mathfrak{z}_{g_0}$ in $\mathbf{H}(\mathbb{A}_f)$ is

$$\mathrm{Stab}_{\mathbf{H}(\mathbb{A}_f)}\mathfrak{z}_{g_0} = \left(Z_{\mathbf{G}}(\mathbb{Q}) \cdot \mathbf{H}(\mathbb{Q}) \cdot g_0 K g_0^{-1}\right) \cap \mathbf{H}(\mathbb{A}_f).$$

**Lemma** VII.1.3.1. *We may rewrite the stabilizer as*

$$\mathit{Stab}_{\mathbf{H}(\mathbb{A}_f)}\mathfrak{z}_{g_0} = \mathbf{H}(\mathbb{Q}) \cdot \left(K^Z_{\mathbf{H},g_0,S} \times K^S_{\mathbf{H}}\right),$$

*where* $K^Z_{\mathbf{H},g_0,S} := \left((Z_{\mathbf{G}}(\mathbb{Q}) \cap K^S) \cdot g_{0,S} K_S g_{0,S}^{-1}\right) \cap \mathbf{H}(F_S)$[4].

*Proof.* We have

$$\begin{aligned}
\mathrm{Stab}_{\mathbf{H}(\mathbb{A}_f)}\mathfrak{z}_{g_0} &\overset{(0)}{=} \mathbf{H}(\mathbb{Q}) \cdot \left((Z_{\mathbf{G}}(\mathbb{Q}) \cap Z_{\mathbf{H}}(\mathbb{Q})K^S) \cdot g_0 K g_0^{-1} \cap \mathbf{H}(\mathbb{A}_f)\right) \\
&\overset{(1)}{=} \mathbf{H}(\mathbb{Q}) \cdot \left((Z_{\mathbf{G}}(\mathbb{Q}) \cap K^S) \cdot g_0 K g_0^{-1} \cap \mathbf{H}(\mathbb{A}_f)\right) \\
&\overset{(2)}{=} \mathbf{H}(\mathbb{Q}) \cdot \left(\left((Z_{\mathbf{G}}(\mathbb{Q}) \cap K^S) \cdot g_{0,S} K_S g_{0,S}^{-1}\right) \cap \mathbf{H}(F_S) \times K^S_{\mathbf{H}}\right) \\
&= \mathbf{H}(\mathbb{Q}) \cdot \left(K^Z_{\mathbf{H},g_0,S} \times K^S_{\mathbf{H}}\right),
\end{aligned}$$

where, (0) is a straightforward consequence of the fact that $g_0^S = 1 \in \underline{\mathbf{G}}(\mathbb{A}^S_{F,f})$ and $K^S_{\mathbf{H}} = K^S \cap \underline{\mathbf{H}}(\mathbb{A}^S_{F,f})$.

(1) Let $z_G = z_H k \in Z_{\mathbf{G}}(\mathbb{Q}) \cap Z_{\mathbf{H}}(\mathbb{Q})K^S = Z_{\mathbf{G}}(\mathbb{Q}) \cap Z_{\mathbf{H}}(\mathbb{Q})\underline{\mathbf{G}}(\mathcal{O}^S_F)$. Write $z_G = (z_V, z_W)$ and $z_H k = \Delta(z'_W)(k_V, k_W) = k z_H$, hence[5]

$$z_V w_{n+1} = k_V z'_W w_{n+1} = k_V w_{n+1},$$

so $z_V \in \underline{\mathbf{T}}^1(\mathcal{O}^S_F) = Z_{\underline{\mathbf{U}}_V}(\mathcal{O}^S_F)$. Accordingly $z'_W \in \underline{\mathbf{T}}^1(\mathcal{O}^S_F) = Z_{\underline{\mathbf{U}}_W}(\mathcal{O}^S_F)$ and $z_G \in K^S$, i.e. $Z_{\mathbf{G}}(\mathbb{Q}) \cap Z_{\mathbf{H}}(\mathbb{Q})K^S = Z_{\mathbf{G}}(\mathbb{Q}) \cap K^S$.

(2) Let $(z g_{0,S} k_S g_{0,S}^{-1}, k^S) \in K^Z_{\mathbf{H},g_0,S} \times K^S_{\mathbf{H}} \subset \underline{\mathbf{H}}(\mathbb{A}_{F,f})$, for some $z \in Z_{\mathbf{G}}(\mathbb{Q}) \cap K^S$. Hence

$$\begin{aligned}
(z g_{0,S} k_S g_{0,S}^{-1}, k^S) &= z(g_{0,S} k_S g_{0,S}^{-1}, \underbrace{z^{-1} k^S}_{:= k'^S \in K^S}) \\
&= z(g_{0,S} k_S g_{0,S}^{-1}, k'^S) \\
&= z g_0 (k_S, k'^S) g_0^{-1} \in (Z_{\mathbf{G}}(\mathbb{Q}) \cap K^S) g_0 K g_0^{-1} \cap \mathbf{H}(\mathbb{A}_f),
\end{aligned}$$

and so $((Z_{\mathbf{G}}(\mathbb{Q}) \cap K^S) \cdot g_0 K g_0^{-1}) \cap \mathbf{H}(\mathbb{A}_f) = K^Z_{\mathbf{H},g_0,S} \times K^S_{\mathbf{H}}$. □

Set $\mathcal{K} := E'_{g_0}$, the field of definition of the cycle $\mathfrak{z}_{g_0}$, it is the subfield of $E_{g_0} \subset E(\infty)$[6] fixed

[4]Note that by SV5 of Remark VI.7.0.1 the subgroup $Z_{\mathbf{G}}(\mathbb{Q})$ is discrete in $Z_{\mathbf{G}}(\mathbb{A}_f)$ and consequently the intersection $Z_{\mathbf{G}}(\mathbb{Q}) \cap K^S$ must be finite.

[5]Recall that $w_{n+1}$ is the fixed generator of the global $E$-hermitian line $D$ which is orthogonal to $W$.

[6]Recall that $K_{\mathbf{H},g_0} = g_0 K g_0^{-1} \cap \mathbf{H}(\mathbb{A}_f)$ and $E_{g_0} \subset E(\infty)$ is the subfield fixed by $\mathrm{Art}^1_E\left(\mathbf{T}^1(\mathbb{Q})\det(K_{\mathbf{H},g_0})\right)$.

by (see §VI.15)

$$\mathrm{Art}_E^1\left(\det\left(\mathbf{H}(\mathbb{Q})\cdot(K^Z_{\mathbf{H},g_0,S}\times K^S_{\mathbf{H}})\right)\right)=\mathrm{Art}_E^1\left(\mathbf{T}^1(\mathbb{Q})\det\left(K^Z_{\mathbf{H},g_0,S}\times K^S_{\mathbf{H}}\right)\right),$$

where, $\mathrm{Art}_E^1$ is the map defined above Corollary VI.14.3.1. Therefore,

$$\frac{\mathbb{A}^\times_{E,f}}{E^\times\mathbb{A}^\times_{F,f}(\mathcal{O}^\times_{g_0,S}\times\widehat{\mathcal{O}^S_E}^\times)}\xrightarrow[\simeq]{\underline{\nu}}\frac{\mathbf{T}^1(\mathbb{A}_f)}{\mathbf{T}^1(\mathbb{Q})(U_{g_0,S}\times U^S)}\xrightarrow[\simeq]{\mathrm{Art}_E^1}\mathrm{Gal}(\mathcal{K}/E)$$

where, $U_{g_0,S}:=\det K^Z_{\mathbf{H},g_0,S}$, $U^S:=\det K^S_{\mathbf{H}}$, and $\mathcal{O}^\times_{g_0,S}\subset(E\otimes F_S)^\times$ such that $\underline{\nu}(\mathcal{O}^\times_{g_0,S})=U_{g_0,S}$ and finally $\widehat{\mathcal{O}^S_E}=\mathcal{O}_E\otimes_{\mathcal{O}^S_F}\widehat{\mathcal{O}^S_F}=\prod_{v\notin S}\mathcal{O}^\times_{E_v}\subset\mathbb{A}^S_{E,f}$ (see Remark VII.1.2.2), where we have denoted, slightly abusively, $\mathcal{O}_{E_v}=\mathcal{O}_E\otimes_{\mathcal{O}_F}\mathcal{O}_{F_v}$ the maximal order of the quadratic étale algebra $E_v=E\otimes_F F_v$.

### VII.1.4 Ramification in transfer fields extensions

Let $v$ be a finite place of $F$ that is unramified in the extension $E/F$, hence

$$E_v^\times/F_v^\times\mathcal{O}^\times_{E_v}\simeq\begin{cases}\varpi_v^{\mathbb{Z}} & \text{If } v \text{ split in } E,\\ \{1\} & \text{If } v \text{ is inert in } E.\end{cases}$$

**Lemma** VII.1.4.1. *Let $\mathfrak{c}\subset\mathcal{O}_F$ be any non-zero ideal of $\mathcal{O}_F$. Any prime of $E$ not dividing $\mathfrak{c}\mathcal{O}_E$ is unramified in $E[\mathfrak{c}]/E$, hence also in $E(\mathfrak{c})/E$.*

*Proof.* Let $v$ be any place of $E$, and $E_v$ the completion of $E$ at the place $v$. The extension $E[\mathfrak{c}]/E$ being Galois, then the various completions $E[\mathfrak{c}]_w$ with $w$ a place of $E[\mathfrak{c}]$ extending $v$ are isomorphic, let $E[\mathfrak{c}]_w$ be (any) one of these completions. Let $\mathfrak{q}$ be the prime of $E[\mathfrak{c}]$ corresponding to the place $w$ and identify the local Galois group $\mathrm{Gal}(E[\mathfrak{c}]_w/E_v)$ with the decomposition group of $D_\mathfrak{q}(E[\mathfrak{c}]/E)\subset\mathrm{Gal}(E[\mathfrak{c}]/E)$ (the stabilizer of $\mathfrak{q}$ in $\mathrm{Gal}(E[\mathfrak{c}]/E)$). The decomposition group $D_\mathfrak{q}(E[\mathfrak{c}]/E)$ is the image of the composition of the following two maps

$$E_v^\times\hookrightarrow\mathbb{A}^\times_{E,f}\twoheadrightarrow\mathbb{A}^\times_{E,f}/E^\times\widehat{\mathcal{O}}^\times_\mathfrak{c}\xrightarrow{\simeq}\mathrm{Gal}(E[\mathfrak{c}]/E),$$

and the image of $\mathcal{O}^\times_{E,v}$ in $\mathrm{Gal}(E[\mathfrak{c}]/E)$ is precisely the inertia group $I(E[\mathfrak{c}]/E)$. Now, because $v\nmid\mathfrak{c}\mathcal{O}_E$ we see that the image of $\mathcal{O}^\times_{E,v}$ is trivial, i.e. $v$ is unramified in the extension $E[\mathfrak{c}]/E$. $\square$

### VII.1.5 Ramification in $\mathcal{K}$-transfer fields extensions

For any nonzero ideal $\mathfrak{f} \subset \mathcal{O}_F$ that is prime to $S$, we consider the field $\mathcal{K}(\mathfrak{f}) \subset E(\infty)$, fixed by $\mathrm{Art}_E^1(U_\mathfrak{f})$, where $U_\mathfrak{f} := U_{g_0,S} \times U_\mathfrak{f}^S$ with $U_\mathfrak{f}^S = \prod_{v \notin S} U_v^1(\mathrm{ord}_{F_v}(\mathfrak{f}))$ and

$$U_v^1(c) := \underline{\nu}\big(\underbrace{(\mathcal{O}_{F_v} + \varpi_v^c \mathcal{O}_{E_v})}_{:=\mathcal{O}_{v,c}}{}^\times\big), \quad v \notin S,\, c \in \mathbb{N}.$$

The fields $\mathcal{K}(\mathfrak{f})$ will be called the $\mathcal{K}$-transfer field of conductors $\mathfrak{f}$. Set

$$\mathfrak{O}_\mathfrak{f} := \mathcal{O}_{g_0,S} \times (\mathcal{O}_\mathfrak{f} \otimes_{\mathcal{O}_F^S} \widehat{\mathcal{O}_F^S}) = \mathcal{O}_{g_0,S} \times \prod_{v \notin S} \mathcal{O}_{v,\mathrm{ord}_{F_v}(\mathfrak{f})} \subset \widehat{\mathcal{O}_E},$$

in particular, we have $\underline{\nu}(\mathfrak{O}_\mathfrak{f}^\times) = U_\mathfrak{f}$. Note that $\mathcal{K}(1) = \mathcal{K}$. Moreover, for every two $\mathcal{O}_F$-ideals $\mathfrak{n}, \mathfrak{f} \subset \mathcal{O}_F$ prime to $S$, we have $\mathcal{K}(\mathfrak{f}\mathfrak{n}) \supset \mathcal{K}(\mathfrak{f})$ and isomorphisms:

$$\frac{E^\times \mathbb{A}_{F,f}^\times \mathfrak{O}_\mathfrak{f}^\times}{E^\times \mathbb{A}_{F,f}^\times \mathfrak{O}_{\mathfrak{f}\mathfrak{n}}^\times} \xrightarrow[\simeq]{\underline{\nu}} \frac{\mathbf{T}^1(\mathbb{Q})U_\mathfrak{f}}{\mathbf{T}^1(\mathbb{Q})U_{\mathfrak{f}\mathfrak{n}}} \xrightarrow[\simeq]{\mathrm{Art}_E^1} \mathrm{Gal}(\mathcal{K}(\mathfrak{f}\mathfrak{n})/\mathcal{K}(\mathfrak{f})).$$

We then obtain an exact sequence[7]

$$1 \longrightarrow \frac{E^\times \cap \mathbb{A}_{F,f}^\times \mathfrak{O}_\mathfrak{f}^\times}{E^\times \cap \mathbb{A}_{F,f}^\times \mathfrak{O}_{\mathfrak{f}\mathfrak{n}}^\times} \longrightarrow \frac{\mathfrak{O}_\mathfrak{f}^\times}{\mathfrak{O}_{\mathfrak{f}\mathfrak{n}}^\times} \longrightarrow \mathrm{Gal}(\mathcal{K}(\mathfrak{f}\mathfrak{n})/\mathcal{K}(\mathfrak{f})) \longrightarrow 1 \tag{VII.2}$$

with

$$\frac{\mathfrak{O}_\mathfrak{f}^\times}{\mathfrak{O}_{\mathfrak{f}\mathfrak{n}}^\times} \simeq \frac{(\mathfrak{O}_\mathfrak{f}^S)^\times}{(\mathfrak{O}_{\mathfrak{f}\mathfrak{n}}^S)^\times} \simeq \prod_{v:\, \mathfrak{p}_v | \mathfrak{n}} \frac{\mathcal{O}_{v,\mathrm{ord}_{F_v}(\mathfrak{f})}^\times}{\mathcal{O}_{v,\mathrm{ord}_{F_v}(\mathfrak{f}\mathfrak{n})}^\times}.$$

Moreover, one can explicitly describe the left global error term appearing in the above exact sequence:

**Lemma** VII.1.5.1. *The natural inclusion map yields an inclusion*

$$\frac{\mathcal{O}_E^\times \cap \mathfrak{O}_\mathfrak{f}^\times}{\mathcal{O}_E^\times \cap \mathfrak{O}_{\mathfrak{f}\mathfrak{n}}^\times} \hookrightarrow \frac{E^\times \cap \mathbb{A}_{F,f}^\times \mathfrak{O}_\mathfrak{f}^\times}{E^\times \cap \mathbb{A}_{F,f}^\times \mathfrak{O}_{\mathfrak{f}\mathfrak{n}}^\times},$$

*with finite cokernel of size smaller than* 2.

*Proof.* This is [Nek07, Proposition (2.9)]. □

For later use, we set $d_\mathfrak{f} := \#\left(\frac{E^\times \cap \mathbb{A}_{F,f}^\times \mathcal{O}_1^\times}{E^\times \cap \mathbb{A}_{F,f}^\times \mathfrak{O}_\mathfrak{f}^\times}\right)$ for any $\mathcal{O}_F$-ideal $\mathfrak{f}$.

**Remark** VII.1.5.1. *Observe that by (VI.6), for any ideal $\mathfrak{f} \subset \mathcal{O}_F$ prime to $S$, there exists a smallest non-zero ideal $\mathfrak{c}_\mathfrak{f} \subset \mathcal{O}_F$ with respect to divisibility such that*

$$\widehat{\mathcal{O}}_{\mathfrak{c}_\mathfrak{f}} \subset \mathfrak{O}_\mathfrak{f} \subset \widehat{\mathcal{O}}_\mathfrak{f} \subset \widehat{\mathcal{O}}_E$$

[7] This is a consequence of the following elementary fact: if one has three subgroups $A$ and $C \subset B$ of some abelian group, then the inclusion maps yield an exact sequence

$$1 \longrightarrow \frac{A \cap B}{A \cap C} \longrightarrow \frac{B}{C} \longrightarrow \frac{AB}{AC} \longrightarrow 1.$$

*Equivalently, $E(\mathfrak{c}_{\mathfrak{f}})$ is the smallest transfer field containing $\mathcal{K}(\mathfrak{f})$:*

$$E(\mathfrak{f}) \subset \mathcal{K}(\mathfrak{f}) \subset E(\mathfrak{c}_{\mathfrak{f}}). \quad \square$$

**Lemma** VII.1.5.2. *For any ideal $\mathfrak{f} \subset \mathcal{O}_F$ prime to $S$, the field $\mathcal{K}(\mathfrak{f})$ is contained in the transfer field $E(\mathfrak{c}_1\mathfrak{f})$.*

*Proof.* We have $\mathfrak{O}_1 \cap \widehat{\mathcal{O}_{\mathfrak{f}}} = \mathfrak{O}_{\mathfrak{f}}$, but $\mathfrak{O}_1 \supset \widehat{\mathcal{O}}_{\mathfrak{c}_1}$ (Remark VII.1.5.1). It follows that

$$\mathfrak{O}_{\mathfrak{f}} \supset \widehat{\mathcal{O}}_{\mathfrak{c}_1} \cap \widehat{\mathcal{O}_{\mathfrak{f}}} \supset \widehat{\mathcal{O}}_{\mathfrak{c}_1\mathfrak{f}},$$

hence $E(\mathfrak{c}_1\mathfrak{f}) \supset \mathcal{K}(\mathfrak{f})$. $\square$

Using Lemmas VII.1.4.1 and VII.1.5.2 we get the following immediate consequence:

**Corollary** VII.1.5.1. *Let $\mathfrak{f} \subset \mathcal{O}_F$ be a non-zero ideal of $\mathcal{O}_F$ prime to $S$. If $\mathfrak{p}$ is a prime ideal of $\mathcal{O}_F$ not dividing $\mathfrak{c}_1\mathfrak{f}$ then each prime of $E$ above $\mathfrak{p}$ is unramified in $\mathcal{K}(\mathfrak{f})$.*

**Lemma** VII.1.5.3. *Let $\mathfrak{f}$ be any non-zero ideal of $\mathcal{O}_F$ satisfying*

$$\mathfrak{f} \nmid I_0 := lcm\{(u-1)\colon u \in (\mathcal{O}_E^\times)_{tors}, u \neq 1\}.$$

*For any ideal $\mathfrak{n} \subset \mathcal{O}_F$, we have*

$$E^\times \cap \mathbb{A}_{F,f}^\times \mathfrak{O}_{\mathfrak{n}\mathfrak{f}}^\times \simeq F^\times \quad \mathcal{O}_E^\times \cap \mathfrak{O}_{\mathfrak{n}\mathfrak{f}}^\times \simeq \mathcal{O}_F^\times.$$

*Proof.* This is [Nek07, Proposition 2.10]. $\square$

Applying Lemma VII.1.5.3 to (VII.2) yields an isomorphism of groups:

**Corollary** VII.1.5.2. *Let $\mathfrak{f}$ be any non-zero ideal of $\mathcal{O}_F$ prime to $S$ and not dividing $I_0$. For any $\mathcal{O}_F$-ideal $\mathfrak{n} \subset \mathcal{O}_F$, we have:*

$$Gal(\mathcal{K}(\mathfrak{n}\mathfrak{f})/\mathcal{K}(\mathfrak{f})) \simeq \frac{\mathfrak{O}_{\mathfrak{f}}^\times}{\mathfrak{O}_{\mathfrak{n}\mathfrak{f}}^\times} \simeq \prod_{v \in Spec(\mathcal{O}_F)\colon \mathfrak{p}_v | \mathfrak{n}} \frac{\mathcal{O}^\times_{v,\mathrm{ord}_{F_v}(\mathfrak{f})}}{\mathcal{O}^\times_{v,\mathrm{ord}_{F_v}(\mathfrak{f}\mathfrak{n})}}.$$

### VII.1.6 Interlude on orders

Define the Artin symbol,

$$\left(\frac{\mathfrak{d}_E}{\mathfrak{p}_v}\right) = \begin{cases} -1 & \text{if } \mathfrak{p}_v \text{ remains inert in } E, \\ 1 & \text{if } \mathfrak{p}_v \text{ splits in } E, \end{cases}$$

where $\mathfrak{d}_E$ is the different ideal of $E$. For $\mathfrak{p}_v \in \mathrm{Spec}(\mathcal{O}_F)$, with $v \notin S$ (in particular unramified in $E/F$), set $\mathbb{F}^k(v) := \mathcal{O}_{F_v}/\mathfrak{p}_v^k$. When $k = 1$ this is the residue field of $\mathcal{O}_{F_v}$

whose size is $q_v := \#\mathbb{F}^1(v)$. For any integer $k \geq 1$, set $\mathbb{F}^k(v)[\epsilon] := \mathbb{F}^k(v)[X]/\langle X^2 \rangle$ to be the infinitesimal deformation $\mathbb{F}^k(v)$-algebra and set $\mathbb{E}^k(v) := \mathcal{O}_{E_v}/\mathfrak{p}_v^k \mathcal{O}_{E_v}$. We then have ring isomorphisms

$$\mathbb{E}^1(v) \simeq \begin{cases} \text{a quadratic extension of } \mathbb{F}^1(v) & \text{if } \left(\frac{\mathfrak{d}_E}{\mathfrak{p}_v}\right) = -1, \\ \mathbb{F}^1(v) \oplus \mathbb{F}^1(v) & \text{if } \left(\frac{\mathfrak{d}_E}{\mathfrak{p}_v}\right) = 1. \end{cases}$$

Here, we summarize a few facts on $\mathcal{O}_{F_v}$-orders of $E_v$.The map that sends an ideal $\mathfrak{p}_v^{c_v} \subset \mathcal{O}_{F_v}$ to the order $\mathcal{O}_{v,c_v} = \mathcal{O}_{F_v} + \mathfrak{p}_v^{c_v}\mathcal{O}_{E_v}$ induces a bijection between the set of ideals of $\mathcal{O}_{F_v}$ and the set of $\mathcal{O}_{F_v}$-orders in $E_v$. These orders are all Gorenstein and local whenever $c_v \geq 1$ with maximal ideal $\mathfrak{P}_{v,c_v} := \mathfrak{p}_v \mathcal{O}_{v,c_v-1}$ and

$$\mathcal{O}_{v,c_v}/\mathfrak{P}_{v,c_v} \simeq \mathbb{F}^1(v).$$

When $c_v = 0$, the order $\mathcal{O}_{v,0}$ is only semi-local if $v$ splits in $E$, since $\mathcal{O}_{E_v} \simeq \mathcal{O}_{F_v} \oplus \mathcal{O}_{F_v}$.

Let $\mathrm{Tr}\colon \mathcal{O}_{E_v} \to \mathcal{O}_{F_v}$ be the usual trace map $z \mapsto z + \overline{z}$. Let $\alpha_v \in \mathcal{O}_{E_v}^\times$ be any generator of the rank 1 $\mathcal{O}_{F_v}$-module $\ker \mathrm{Tr}$[8], therefore $\overline{\alpha}_v = -\alpha_v, \alpha_v^2 \in \mathcal{O}_{F_v}$ and for every $c_v \geq 0$ we have $\mathcal{O}_{v,c_v} = \mathcal{O}_{F_v} \oplus \mathfrak{p}_v^{c_v}\alpha_v$.

**Lemma** VII.1.6.1. *Let $c > 0$, we have an isomorphism of groups*

$$\frac{\mathcal{O}_{v,0}^\times}{\mathcal{O}_{v,c}^\times} \simeq \mathbb{E}^c(v)^\times/\mathbb{F}^c(v)^\times.$$

*Proof.* Consider the following composition of reduction maps

$$\mathcal{O}_{E_v} = \mathcal{O}_{v,0} \twoheadrightarrow \mathbb{E}^k(v)^\times \twoheadrightarrow \mathbb{E}^k(v)^\times/\mathbb{F}^k(v)^\times.$$

The quotient map of rings $\mathcal{O}_{v,0} \twoheadrightarrow \mathcal{O}_{v,0}/\mathfrak{p}_v^c\mathcal{O}_{v,0}$ induces a surjective homomorphism of groups $\mathcal{O}_{v,0}^\times \twoheadrightarrow (\mathcal{O}_{v,0}/\mathfrak{p}_v^c\mathcal{O}_{v,0})^\times$ with kernel $1 + \mathfrak{p}_v^c\mathcal{O}_{v,0}$, i.e.

$$\mathcal{O}_{v,0}^\times/1 + \mathfrak{p}_v^c\mathcal{O}_{v,0} \simeq (\mathcal{O}_{v,0}/\mathfrak{p}_v^c\mathcal{O}_{v,0})^\times.$$

But since the diagonal image of $\mathcal{O}_F \to \mathcal{O}_{v,0}/\mathfrak{p}_v^c\mathcal{O}_{v,0}$ is precisely $\mathbb{F}^k(v)$, we deduce that the kernel of the following composition of reduction maps

$$f\colon \mathcal{O}_{E_v} = \mathcal{O}_{v,0} \twoheadrightarrow \mathbb{E}^k(v)^\times \twoheadrightarrow \mathbb{E}^k(v)^\times/\mathbb{F}^k(v)^\times.$$

is precisely

$$\mathcal{O}_F^\times(1 + \mathfrak{p}_v^c\mathcal{O}_{v,0}) = \mathcal{O}_{v,c}^\times. \quad \square$$

[8]For example, if $v$ split then one has a decomposition $\mathcal{O}_{E_v} \simeq \mathcal{O}_{F_v} \oplus \mathcal{O}_{F_v}$, such that $\tau \in \mathrm{Gal}(E/F)$ (or just the complex conjugation since we have identified $E$ with $\iota_1(E)$) acts by swapping the components. Therefore, $\ker \mathrm{Tr} = \mathcal{O}_{F_v} \cdot (1,-1)$ and one can take $\alpha_v = (1,-1)$, in addition, observe that $\overline{(1,-1)} = -(1,1)$ and $(1,-1)^2 = (1,1) \in \mathcal{O}_{F_v}$ where $\mathcal{O}_{F_v}$ is embedded diagonally in $\mathcal{O}_{E_v}$.

**Lemma** VII.1.6.2. *Let $c \geq k > 0$, we have an isomorphism*

$$\mathcal{O}_{v,c}^{\times}/\mathcal{O}_{v,c+k}^{\times} \simeq \left(\mathbb{F}^k(v)[\epsilon]\right)^{\times}/\mathbb{F}^k(v)^{\times}.$$

*Proof.* As in the previous proof, consider the quotient map of rings $\mathcal{O}_{v,c} \twoheadrightarrow \mathcal{O}_{v,c}/\mathfrak{p}_v^k\mathcal{O}_{v,c}$. It induces a surjective homomorphism of groups $\mathcal{O}_{v,c}^{\times} \twoheadrightarrow \left(\mathcal{O}_{v,c}/\mathfrak{p}_v^k\mathcal{O}_{v,c}\right)^{\times}$ with kernel $1+\mathfrak{p}_v^k\mathcal{O}_{v,c}$, i.e.

$$\mathcal{O}_{v,c}^{\times}/1+\mathfrak{p}_v^k\mathcal{O}_{v,c} \simeq \left(\mathcal{O}_{v,c}/\mathfrak{p}_v^k\mathcal{O}_{v,c}\right)^{\times}.$$

We may also consider the map of rings $\mathcal{O}_{v,c+k} \twoheadrightarrow \mathcal{O}_{v,c+k}/\mathfrak{p}_v^k\mathcal{O}_{v,c}$. It is clearly surjective and induces the isomorphism of groups

$$\mathcal{O}_{v,c+k}^{\times}/\left(1+\mathfrak{p}_v^k\mathcal{O}_{v,c}\right) \simeq \left(\mathcal{O}_{v,c+k}/\mathfrak{p}_v^k\mathcal{O}_{v,c}\right)^{\times}.$$

Therefore,

$$\begin{aligned}\mathcal{O}_{v,c}^{\times}/\mathcal{O}_{v,c+k}^{\times} &\simeq \mathcal{O}_{v,c}^{\times}/(1+\mathfrak{p}_v^k\mathcal{O}_{v,c})/\mathcal{O}_{v,c+k}^{\times}/(1+\mathfrak{p}_v^k\mathcal{O}_{v,c})\\ &\simeq \left(\mathcal{O}_{v,c}/\mathfrak{p}_v^k\mathcal{O}_{v,c}\right)^{\times}/\left(\mathcal{O}_{v,c+k}/\mathfrak{p}_v^k\mathcal{O}_{v,c}\right)^{\times}\\ &\simeq \left(\mathcal{O}_{v,c}/\mathfrak{p}_v^k\mathcal{O}_{v,c}\right)^{\times}/\left(\mathcal{O}_{F_v}/\mathfrak{p}_v^k\mathcal{O}_{F_v}\right)^{\times}\end{aligned}$$

Recall $\mathbb{F}^k(v) = \mathcal{O}_{F_v}/\mathfrak{p}_v^k\mathcal{O}_{F_v}$, and consider the homomorphism of rings $\mathbb{F}^k(v)[X] \to \mathcal{O}_{v,c}/\mathfrak{p}_v^k\mathcal{O}_{v,c}$, given by $X \mapsto (\varpi_v^c\alpha_v \mod \mathfrak{p}_v^k\mathcal{O}_{v,c})$. The kernel contains $\langle X^2\rangle$ (because $c \geq k$) and gives a surjective map between two sets with the same order. Hence, for any $c \geq k > 0$ that

$$\mathcal{O}_{v,c}^{\times}/\mathcal{O}_{v,c+k}^{\times} \simeq \left(\mathbb{F}^k(v)[\epsilon]\right)^{\times}/\mathbb{F}^k(v)^{\times}. \quad \square$$

In particular, $\mathbb{G}_v := \mathbb{E}^1(v)^{\times}/\mathbb{F}^1(v)^{\times} \simeq \mathcal{O}_{v,0}^{\times}/\mathcal{O}_{v,1}^{\times}$ is (cyclic if $v$ is inert in $E/F$) of order $q_v - \left(\frac{\mathfrak{d}_E}{\mathfrak{p}_v}\right)$, and $\mathbb{G}_v(\epsilon) := \mathbb{F}^1(v)[\epsilon]^{\times}/\mathbb{F}^1(v)^{\times} \simeq \mathcal{O}_{v,c}^{\times}/\mathcal{O}_{v,c+1}^{\times}$. Finally, for every $c > 0$, we have the following short exact sequence (of abelian groups)

$$1 \longrightarrow \frac{\mathcal{O}_{v,1}^{\times}}{\mathcal{O}_{v,c}^{\times}} \longrightarrow \frac{\mathcal{O}_{v,0}^{\times}}{\mathcal{O}_{v,c}^{\times}} \longrightarrow \frac{\mathcal{O}_{v,0}^{\times}}{\mathcal{O}_{v,1}^{\times}} \longrightarrow 1,$$

thus

$$\#\left(\frac{\mathcal{O}_{v,0}^{\times}}{\mathcal{O}_{v,c}^{\times}}\right) = |\mathbb{G}_v(\epsilon)|^{c-1}\cdot|\mathbb{G}_v| = q_v^{c-1}\left(q_v - \left(\frac{\mathfrak{d}_E}{\mathfrak{p}_v}\right)\right).$$

### VII.1.7 Galois groups

**Definition** VII.1.7.1. *Set*

$$\mathcal{P} := \{\mathfrak{p} \in Spec(\mathcal{O}_F)\colon \mathfrak{p} \textit{ is unramified in } E/F, \mathfrak{p} \notin S, \mathfrak{p} \nmid \mathfrak{c}_1, \mathfrak{p}\mathcal{O}_E \nmid I_0\},$$

*and let $\mathcal{P}_{sp}$, (resp. $\mathcal{P}_{in}$) denotes the subset of $\mathcal{P}$ of prime ideals of $F$ that are split, (resp. inert) in $E/F$.*

*Denote by $\mathcal{N} = \bigcup_{r\geq 1} \mathcal{N}^r = \bigcup_{r\geq 1} \cup_{s=1}^{s=r} \left(\mathcal{N}_{sp}^s \cdot \mathcal{N}_{in}^{r-s}\right)$ the set of square-free products of ideals in $\mathcal{P}$:*

$$\forall r \geq 1\colon \mathcal{N}_?^r := \{\mathfrak{p}_1 \cdots \mathfrak{p}_r \colon \mathfrak{p}_j \in \mathcal{P}_? \text{ distinct}\} \text{ for } ? \in \{in,\ sp\}.$$

So in particular, one has:

**Proposition** VII.1.7.1. *Let $\mathfrak{f} = \prod_i \mathfrak{p}_i \in \mathcal{N}^r$, and $\mathfrak{p} \in \mathcal{P}$ prime to $\mathfrak{f}$, i.e. $\mathfrak{p}\mathfrak{f} \in \mathcal{N}^{r+1}$.*

*(i) The extension $\mathcal{K}(\mathfrak{p}\mathfrak{f})/\mathcal{K}(\mathfrak{f})$ is of degree $(q_v - \left(\frac{\mathfrak{d}_E}{\mathfrak{p}_v}\right))/u(r)$, where*

$$u(0) = [E^\times \cap \mathbb{A}_{F,f}^\times \mathfrak{O}_1 : F^\times] = c[\mathcal{O}_E^\times \cap \mathfrak{O}_1 : \mathcal{O}_F^\times], \text{ with } c \in \{1, 2\}$$

*and $u(r) = 1$ if $r \geq 1$.*

*(ii) Define*

$$\mathcal{K}(\mathfrak{f})' = \mathcal{K}(\mathfrak{p}_1) \cdots \mathcal{K}(\mathfrak{p}_r),$$

*with $\mathcal{K}(1)' = \mathcal{K}$ and set $G(\mathfrak{f}) := Gal(\mathcal{K}(\mathfrak{f})'/\mathcal{K})$. The canonical map*

$$G(\mathfrak{f}) \longrightarrow G(\mathfrak{p}_1) \times \cdots \times G(\mathfrak{p}_r),$$

*is an isomorphism and we have $[\mathcal{K}(\mathfrak{f}) : \mathcal{K}(\mathfrak{f})'] = u(r)u(0)^{r-1}$.*

*Proof.* (i) If $r \geq 1$ this is a special case of Corollary VII.1.5.2, which says

$$\mathrm{Gal}(\mathcal{K}(\mathfrak{p}\mathfrak{f})/\mathcal{K}(\mathfrak{f})) \simeq \mathbb{G}_v,$$

thus of order $q_v - \left(\frac{\mathfrak{d}_E}{\mathfrak{p}_v}\right)$. If $r = 0$, recall that by (VII.2) and since $\mathfrak{p}\mathcal{O}_E \nmid I_0$, we have an exact sequence

$$\frac{E^\times \cap \mathbb{A}_{F,f}^\times \mathfrak{O}_1^\times}{F^\times} \hookrightarrow \frac{\mathfrak{O}_1^\times}{\mathfrak{O}_\mathfrak{p}^\times} \simeq \frac{\mathcal{O}_{v_\mathfrak{p},0}^\times}{\mathcal{O}_{v_\mathfrak{p},1}} \twoheadrightarrow \mathrm{Gal}(\mathcal{K}(\mathfrak{p})/\mathcal{K}), \tag{VII.3}$$

hence, the extension $\mathcal{K}(\mathfrak{p})/\mathcal{K}$ is of degree $(q_v - \left(\frac{\mathfrak{d}_E}{\mathfrak{p}_v}\right))/[E^\times \cap \mathbb{A}_{F,f}^\times \mathfrak{O}_1^\times : F^\times]$.

(ii) When $r = 0$ this is trivial. If $r \geq 1$, the same proof of [Nek07, (ii) Proposition 4.10] gives the desired statement, which is derived from Lemma 4.11 in *loc. cit.*[9]. Finally, again by (the proof of) [Nek07, Lemma 4.11] we have for $r \geq 1$

$$[\mathcal{K}(\mathfrak{f}) : \mathcal{K}(\mathfrak{f})'] = \left|E^\times \cap \mathbb{A}_{F,f}^\times \mathfrak{O}_1^\times / F^\times\right|^{r-1} = u(0)^{r-1}. \quad \square$$

[9] applied to (with the notation of *loc. cit.*)

$$G = \mathfrak{O}_1/\mathfrak{O}_\mathfrak{f} = \mathfrak{O}_{\mathfrak{p}_1 \cdots \mathfrak{p}_r}, G_j = \mathfrak{O}_{\frac{\mathfrak{f}}{\mathfrak{p}_j}}/\mathfrak{O}_\mathfrak{f}, \widetilde{G}_j = \mathfrak{O}_{\mathfrak{p}_j} \text{ and } A = E^\times \cap \mathbb{A}_{F,f}^\times \mathfrak{O}_1^\times / F^\times.$$

### VII.1.8 Global to local Galois/Hecke action on cycles

We have a surjection (see §VI.12) of $\underline{\mathbf{T}}^1(\mathbb{A}_f) \times \mathcal{H}_K$-modules

$$\pi_{\mathrm{cyc}} \colon \mathbb{Z}[\mathbf{H}^{\mathrm{der}}(\mathbb{A}_f)\backslash \mathbf{G}(\mathbb{A}_f)/K] \longrightarrow\!\!\!\!\!\rightarrow \mathbb{Z}[\mathcal{Z}_{\mathbf{G},K}(\mathbf{H})].$$

The left-hand-side module is factorizable (see [GH19, Def. 5.8]), i.e.:

$$\mathbb{Z}[\mathbf{H}^{\mathrm{der}}(\mathbb{A}_f)\backslash \mathbf{G}(\mathbb{A}_f)/K] = \mathbb{Z}[\underline{\mathbf{H}}^{\mathrm{der}}(F_S)\backslash \underline{\mathbf{G}}(F_S)/K_S] \otimes {\bigotimes_{v\notin S}}' \mathbb{Z}[\underline{\mathbf{H}}^{\mathrm{der}}(F_v)\backslash \underline{\mathbf{G}}(F_v)/K_v],$$

where, $\otimes'_{v\notin S}$ is the restricted product with respect to the elements[10]

$$\{[1]_v \in \underline{\mathbf{H}}^{\mathrm{der}}(F_v)\backslash \underline{\mathbf{G}}(F_v)/K_v\}_{v\notin S},$$

and the equality above intertwines the action of $\mathcal{H}_K$ (resp. $\underline{\mathbf{T}}^1(\mathbb{A}_f)$) with the action of

$$\mathcal{H}_{K_S} \otimes {\bigotimes_{v\notin S}}' \mathcal{H}_{K_v} \quad (\text{resp. } \underline{\mathbf{T}}^1(F_S) \times {\prod_{v\notin S}}' \underline{\mathbf{T}}^1(F_v)),$$

where, $\mathcal{H}_{K_S} := \mathrm{End}_{\mathbb{Z}[\underline{\mathbf{G}}(F_S)]}\mathbb{Z}[\underline{\mathbf{G}}(F_S)/K_S]$ and $\mathcal{H}_{K_v} = \mathrm{End}_{\mathbb{Z}[\underline{\mathbf{G}}(F_v)]}\mathbb{Z}[\underline{\mathbf{G}}(F_v)/K_v]$, $v \notin S$. where $\otimes'_{v\notin S}$ is the restricted product with respect to

$$\{\mathrm{Id}_{\underline{\mathbf{G}}(F_v)/K_v}\}_{v\notin S} \quad (\text{resp. } \{\underline{\mathbf{T}}^1(\mathcal{O}_{F_v})\}_{v\notin S}).$$

### VII.1.9 Main theorems on distribution

For every place $v$ in $\mathcal{P}_{sp}$ corresponding to the prime ideal $\mathfrak{p}_v \in \mathcal{N}_{sp}$, let $w$ be the place of $E$ defined by the embedding $\iota_v \colon \overline{F} \to \overline{F}_v$ fixed in §VI.1. We denote by $\mathfrak{P}_w$ the prime ideal of $\mathcal{O}_E$ above $\mathfrak{p}_v$ corresponding to the place $w$, and set $\mathrm{Fr}_w$ for the corresponding *geometric* Frobenius[11]. Let $\mathbf{Frob}_w \in \mathbf{T}^1(\mathbb{A}_f)$ be any element such that $\mathrm{Art}^1_E(\mathbf{Frob}_w)|_{E(\infty)^{un,w}} = \mathrm{Fr}_w$, where $E(\infty)^{un,w}$ is the maximal unramified at $w$ extension in $E(\infty)$.

**Theorem** VII.1.9.1. *With the above notation, we have*

$$H_w(\boldsymbol{Frob}_w)([1]_v) \equiv 0 \mod q_v^{n-1}(q_v - 1) \quad \text{in } \mathbb{Z}[q_v^{\pm 1}][\mathbf{H}^{\mathrm{der}}(F_v)\backslash \mathbf{G}(F_v)/K_v],$$

*where $H_w$ is the Hecke polynomial attached to $\mathrm{Sh}_K(\mathbf{G}, \mathcal{X})$ at the place $w$ of the reflex field $E = E(\mathbf{G}, \mathcal{X})$ (see VII.2.3).*

As a corollary of Theorem VII.1.9.1, we obtain local horizontal relations in Corollary VII.2.5.1, from which we derive the tame relations

**Theorem** VII.1.9.2 (Horizontal relations). *Set $\xi_1 = \mathfrak{z}_{g_0}$. There exists a collection of*

[10] The notation $[g]_v$, $(g \in \mathbf{G}(F_v))$, is for $\mathbf{H}^{\mathrm{der}}(F_v) \cdot g \cdot K_v \in \mathbf{H}^{\mathrm{der}}(F_v)\backslash \mathbf{G}(F_v)/K_v$.

[11] Which induces $x \mapsto x^{-q_v}$ on the residue fields of $E_v^{un}$ (resp. $E_w^{un}$ and $E_{\overline{w}}^{un}$).

*cycles* $\xi_{\mathfrak{f}} \subset \mathbb{Z}[q_v^{\pm 1}][\mathcal{Z}_{\mathbf{G},K}(\mathbf{H})]$ *defined over* $\mathcal{K}(\mathfrak{f})$ *(constructed in §VII.3.1) such that for every place* $v \in \mathcal{P}_{sp}$*, with* $\mathfrak{p}_v \nmid \mathfrak{f}$*, we have*

$$H_w(\mathrm{Fr}_w) \cdot \xi_{\mathfrak{f}} = \mathrm{Tr}_{\mathcal{K}(\mathfrak{p}_v\mathfrak{f})/\mathcal{K}(\mathfrak{f})}\, \xi_{\mathfrak{p}_v\mathfrak{f}},$$

*where,* $H_w \in \mathcal{H}_{K_{\mathfrak{p}_v}}(\mathbb{Z}[q_v^{\pm 1}])[X]$ *is the Hecke polynomial attached to* $\mathrm{Sh}_K(\mathbf{G}, \mathcal{X})$ *at the place* $w$ *of the reflex field* $E = E(\mathbf{G}, \mathcal{X})$ *defined by* $\iota_v$*.*

# VII.2 Proof of local distribution relations in the split case

Let $v \in \mathcal{P}_{sp}$. We recall once again that an embedding $\iota_v \colon \overline{F} \hookrightarrow \overline{F}_v$ has been fixed in §VI.1. Let $w$ the place of $E$ above $v$ determined by $\iota_v$, and $\overline{w}$ its conjugate. We abuse notation and also write $w$ for the place above $w$ determined by this choice in any field extension of $E$ contained in the fixed algebraic closure $\overline{F}$. Let $\varpi_v$ be a uniformizer for $F_v = E_w$, $q_v$ for the cardinality of the residue field and $p$ for the rational prime below $v$.

## VII.2.1 Normalization isomorphism in the split case

We identify the group $\mathbf{U}(V)_{/F_v} \times \mathbf{U}(W)_{/F_v}$ with $\mathbf{GL}(V_w)_{/F_v} \times \mathbf{GL}(W_w)_{/F_v}$ as follows: Recall that for any Hermitian $E$-space $\mathcal{V}$

$$\mathbf{U}(\mathcal{V})(F_v) = \{g \in \mathbf{GL}(\mathcal{V})(E \otimes_F F_v) \colon \psi_v(gx, gy) = \psi_v(x, y),\ \forall x, y \in \mathcal{V} \otimes_F F_v\},$$

where $\psi_v = \psi_{F_v}$ (see footnote 1 in on page 161). We have

$$E_v = E \otimes_F F_v = E_w \oplus E_{\overline{w}} \simeq F_v \oplus F_v,$$

where, the action of complex conjugation on the left-hand side corresponds to the involution $(s, t) \mapsto (t, s)$ on the right-hand side. Thus, one has $\mathcal{V} \otimes_F F_v = \mathcal{V}_w \oplus \mathcal{V}_{\overline{w}}$, and

$$\mathbf{GL}(\mathcal{V})(E \otimes_F F_v) = \mathbf{GL}(\mathcal{V}_w) \times \mathbf{GL}(\mathcal{V}_{\overline{w}}) \simeq \mathbf{GL}(\mathcal{V})(F_v) \times \mathbf{GL}(\mathcal{V})(F_v).$$

The hermitian form $\psi_v$ takes values in $E_v = E_w \oplus E_{\overline{w}}$, write $\psi_v = (\psi_w, \psi_{\overline{w}})$ for its two component. For any $x, y \in \mathcal{V} \otimes_F F_v = \mathcal{V}_w \oplus \mathcal{V}_{\overline{w}}$, write $x = x_w + x_{\overline{w}}$ and $y = y_w + y_{\overline{w}}$. Recall that the original hermitian form $\psi$ is semi-linear on the right. By definition

$\psi_v(x,y) = (\psi_w(x,y), \psi_{\overline{w}}(x,y))$, accordingly we have

$$\begin{aligned}
\psi_v(x,y) &= \psi_v\left(x_w, y_{\overline{w}}\right) + \psi_v\left(x_{\overline{w}}, y_w\right) + \psi_v\left(x_w, y_w\right) + \psi_v\left(x_{\overline{w}}, y_{\overline{w}}\right) \\
&\overset{(1)}{=} \psi_v\left(x_w, y_{\overline{w}}\right) + \psi_v\left(x_{\overline{w}}, y_w\right) \\
&= (\psi_w(x_w, y_{\overline{w}}), \psi_{\overline{w}}(x_w, y_{\overline{w}})) + (\psi_w(x_{\overline{w}}, y_w), \psi_{\overline{w}}(x_{\overline{w}}, y_w)) \\
&= (\psi_w(x_w, y_{\overline{w}}), 0) + (0, \psi_{\overline{w}}(x_{\overline{w}}, y_w)) \\
&= (\psi_w(x_w, y_{\overline{w}}), \psi_{\overline{w}}(x_{\overline{w}}, y_w))
\end{aligned}$$

For (1), we have used the fact that $\psi_v = 0$ on $\mathcal{V}_w \times \mathcal{V}_w$ and $\mathcal{V}_{\overline{w}} \times \mathcal{V}_{\overline{w}}$. To show this, let $x, y$ be any elements in $\mathcal{V}_w$ then $x = (1,0)x$ where $(1,0) \in F_v \oplus F_v$, so $\psi_v(x,y) = \psi_v((1,0)x, (1,0)y) = (1,0)(1,0)^{\tau}\psi_v(x,y) = (0,0)\psi_v(x,y) = 0$. We show similarly the annihilation of $\psi_v = 0$ on $\mathcal{V}_{\overline{w}} \times \mathcal{V}_{\overline{w}}$. Accordingly, we have

i. $\psi_w = 0$ on $\mathcal{V}_{\overline{w}} \times \mathcal{V}$ and $\mathcal{V} \times \mathcal{V}_w$, and induces a perfect pairing $\psi_w \colon \mathcal{V}_w \times \mathcal{V}_{\overline{w}} \to F_v$.

ii. $\psi_{\overline{w}} = 0$ on $\mathcal{V}_w \times \mathcal{V}$ and $\mathcal{V} \times \mathcal{V}_{\overline{w}}$, and induces a perfect pairing $\psi_{\overline{w}} \colon \mathcal{V}_{\overline{w}} \times \mathcal{V}_w \to F_v$.

iii. $\psi_{\overline{w}}(x_{\overline{w}}, y_w) = \psi_w(y_w, x_{\overline{w}})$, since $\psi_v(x,y) = \tau(\psi_v(y,x))$.

Let $g = (g_1, g_2) \in \mathbf{GL}(\mathcal{V}_w) \times \mathbf{GL}(\mathcal{V}_{\overline{w}})$

$$\begin{aligned}
\psi_v((gx, gy)) &= \psi_v\left((g_1,g_2)(x_w + x_{\overline{w}}), (g_1,g_2)(y_w + y_{\overline{w}})\right) \\
&= \psi_v\left(g_1x_w + g_2x_{\overline{w}}x, g_1y_w + g_2y_{\overline{w}}\right) \\
&= (\psi_w(g_1x_w, g_2y_{\overline{w}}), \psi_{\overline{w}}(g_2x_{\overline{w}}, g_1y_w)) \\
&= (\psi_w(g_1x_w, g_2y_{\overline{w}}), \psi_w(g_1y_w, g_2x_{\overline{w}}))
\end{aligned}$$

Now, if $g = (g_1, g_2)$ is in $\underline{\mathbf{U}}_{\mathcal{V}}(F_v) \subset \mathbf{GL}(\mathcal{V}_w) \times \mathbf{GL}(\mathcal{V}_{\overline{w}})$, then

$$\psi_v((gx, gy)) = \psi_v(x,y) = (\psi_w(x_w, y_{\overline{w}}), \psi_w(y_w, x_{\overline{w}}))\,.$$

Hence, $g = (g_1, g_2) \in \underline{\mathbf{U}}_{\mathcal{V}}(F_v)$ if and only if $\psi_w(g_1x_w, g_2y_{\overline{w}}) = \psi_w(x_w, y_{\overline{w}})$ for all $x_w \in \mathcal{V}_w$ and $y_{\overline{w}} \in \mathcal{V}_{\overline{w}}$. The discussion above shows that the projection $g = (g_1, g_2) \mapsto g_1$ defines an isomorphism $\underline{\mathbf{U}}(\mathcal{V})(F_v) \simeq \mathbf{GL}(\mathcal{V}_w)$ that is actually defined over $\mathcal{O}_{F_v}$. Now, since $w$ is the place of $E$ corresponding to the fixed embedding $\iota_v \colon \overline{F} \hookrightarrow \overline{F}_v$, there is no ambiguity in writing $\mathcal{V}_w$ as $\mathcal{V}_v$ and viewing it as a vector space over $F_v = E_w$. Therefore, we get the desired identifications.

**Remark** VII.2.1.1. *If we follow Remark VII.1.2.1 and pick up an $\mathcal{O}_E$-lattice $\mathcal{L}_W$ in $W$, $\mathcal{L}_D$ in $D$ and $\mathcal{L}_V = \mathcal{L}_W \oplus \mathcal{L}_D$ in $V$, then working with Hermitian spaces over $\mathcal{O}_{F_v}$ gives an identification:*

$$\underline{\mathbf{U}}_{V,\mathcal{O}_{F_v}} \simeq \mathbf{GL}_{n+1,\mathcal{O}_{F_v}} \quad \textit{and} \quad \underline{\mathbf{U}}_{W,\mathcal{O}_{F_v}} \simeq \mathbf{GL}_{n,\mathcal{O}_{F_v}}$$

*using the previous formulas throughout. We will retain this notation in §VII.2.*

As we have pointed out in §VII.1.6, under the identification $E_v \simeq F_v \oplus F_v$, the maximal order $\mathcal{O}_{E_v}$ of the étale algebra $E_v$ identifies with $\mathcal{O}_{E_v} \simeq \mathcal{O}_{E_w} \oplus \mathcal{O}_{E_{\overline{w}}} \simeq \mathcal{O}_{F_\tau} \oplus \mathcal{O}_{F_\tau}$. The groups

$$U_v^1(c) = \underline{\nu}(\mathcal{O}_{v,c}{}^{\times}) = \underline{\nu}(\mathcal{O}_{F_v} + \varpi_v^c \mathcal{O}_{E_v}{}^{\times}) = \{(z, z^{-1})\colon z \in 1 + \varpi_v^c \mathcal{O}_{F_v}\} \text{ for } c \in \mathbb{N},$$

defined in §VII.1.3, yield the decreasing filtration $(H_c := \det^{-1}(U_v^1(c)))_{c\in\mathbb{N}}$, on $\mathbf{H}(F_v)$. The $E$ determinant map on $\mathbf{U}(\star)$ becomes under the isomorphism $\mathbf{U}(V)_{/F_v} \simeq \mathbf{GL}(\star)_{/F_v}$ the usual determinant map, and the groups $U_v^1(c)$ become $1 + \varpi^c \mathcal{O}_{F_v}$.

By abuse of notation, we will also use the notation $H_c$ for the corresponding subgroup $\det^{-1}(1 + \varpi_v^c \mathcal{O}_{F_v}) \subset \mathbf{GL}_n(F_v)$ via the isomorphism $\mathbf{H}(F_v) \simeq \mathbf{GL}_n(F_v)$ fixed above.

### VII.2.2 Action of Frobenii on unramified special cycles

Let $g \in \mathbf{G}(\mathbb{A}_f)$. Assume that the field of definition of the cycle $\mathfrak{z}_g$ is unramified at $v$. The description of the Galois action on special cycles in §VI.15 using $\mathbf{H}(\mathbb{A}_f)$, together with the discussion in §VII.2.1 imply that the action of $\mathrm{Fr}_w \in \mathrm{Gal}(E_w^{un}/E_w)$ (the geometric Frobenius) on the cycle $\mathfrak{z}_g$ is given by

$$\mathrm{Fr}_w \cdot \mathfrak{z}_g = \mathfrak{z}_{\Delta_v(\mathbf{Frob}_v)\cdot g},$$

where, $\mathbf{Frob}_v \in \mathbf{GL}(W_v) \simeq \mathbf{U}(W)(F_v)$ is any matrix verifying

$$\mathrm{ord}_{F_v}(\det(\mathbf{Frob}_v)) = 1, \text{ e.g. } \mathbf{Frob}_v = \mathrm{diag}(\varpi_v, \cdots, 1),$$

and $\Delta_v$ is the composition of the following natural embeddings:

$$\Delta_v \colon \mathbf{U}(W)(F_v) \hookrightarrow \mathbf{U}(W)(\mathbb{A}_{F,f}) \hookrightarrow \mathbf{U}(V)(\mathbb{A}_{F,f}) \times \mathbf{U}(W)(\mathbb{A}_{F,f}) \simeq \mathbf{G}(\mathbb{A}_f).$$

### VII.2.3 The Hecke polynomial for split places

As usual, to ease notation, let $\star$ denote some/any element in $\{V, W\}$. We continue with the fixed place $v \notin S$ of $F$ that splits in $E$ to $w\overline{w}$.

#### VII.2.3.1 A local pair

The choice of an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{F}_v$ extending the distinguished fixed embedding $\iota_v \colon F_v \to \overline{F}_v$ induces an identification:

$$\begin{array}{ccc} \mathrm{Hom}(F,\mathbb{R}) = \mathrm{Hom}(F,\overline{\mathbb{Q}}) & \xrightarrow{\simeq} & \Sigma_{F,v} = \mathrm{Hom}(F,\overline{F}_v) \\ \widetilde{\iota} & \longmapsto & \iota_v \circ \widetilde{\iota}. \end{array}$$

We have

$$\mathbf{G}_{\star,\overline{F}_v} \simeq \prod_{\widetilde{\iota}\in\Sigma_{F,v}} \mathbf{G}_{\star,\widetilde{\iota}} \quad \text{where} \quad \mathbf{G}_{\star,\widetilde{\iota}} = \mathbf{GL}(\star\otimes\overline{F}_v) \simeq \mathbf{GL}(\dim_E \star)_{\overline{F}_v}.$$

Recall that the conjugacy class of $\mu_h$ with $h = h_{\mathfrak{B}_V} \times h_{\mathfrak{B}_W} \in \mathcal{X}$ is independent of the choice of $h$ and is defined over the reflex field $E$ (§VI.5), hence $[\mu_h] \in \mathcal{M}(E)$. Now using the fixed embedding $E \hookrightarrow \overline{F} \xhookrightarrow{\iota_v} \overline{F}_v$, we get elements $[\mu_{\star,v}] \in \mathcal{M}_{\mathbf{G}_\star}(\overline{F}_v)$ and $[\mu_{h,v}] = [\mu_{V,v} \oplus \mu_{W,v}] \in \mathcal{M}_{\mathbf{G}}(\overline{F}_v)$, where $\mu_{\star,v}$ is given on $\overline{F}_v$-points by

$$\mu_{\star,v} \colon \mathbb{G}_{m,\overline{F}_v} \to \mathbf{G}_{\overline{F}_v} \qquad t \longmapsto \left(\begin{pmatrix} t & \\ & \mathrm{Id}_{\dim_E \star-1}\end{pmatrix}, \mathrm{Id}_{\dim_E \star}, \cdots, \mathrm{Id}_{\dim_E \star}\right).$$

The only nontrivial component is the one corresponding to the distinguished embedding $\iota_1$ under the identification $\mathrm{Hom}(F,\overline{\mathbb{Q}})\simeq\Sigma_{F,v}$. We will then write by abuse of notation $\mu_{h,v} \colon \mathbb{G}_{m,\overline{F}_v} \to \mathbf{U}(V)_{\overline{F}_v} \times \mathbf{U}(W)_{\overline{F}_v}$ given by:

$$t \longmapsto \left(\begin{pmatrix} t & \\ & 1_n\end{pmatrix}, \begin{pmatrix} t & \\ & 1_{n-1}\end{pmatrix}\right) \in \mathbf{T}(\mathfrak{B}_V)(\overline{F}_v) \times \mathbf{T}(\mathfrak{B}_W)(\overline{F}_v),$$

Here, we have used the identification $\mathbf{U}(\star)_{/\overline{F}_v} \simeq \mathbf{GL}(\star_w)_{/\overline{F}_v}$ given by the choice of the place $w$ over $v$, accordingly $\mathbf{T}(\mathfrak{B}_\star)_{\overline{F}_v}$ identifies with the maximal $\overline{F}_v$-torus of diagonal matrices in $\mathbf{GL}(\dim_E \star)_{\overline{F}_v}$ with respect to the local basis $\mathfrak{B}_{\star,v}$ induced from the fixed global basis $\mathfrak{B}_\star$ (§VI.4).

Note that $[\mu_{h,v}]$ is independent of the choice of $\iota_v$ and is invariant under the action of $\mathrm{Gal}(\overline{F}_v/E_w)$ ($E_w = F_v$), thus, by [Kot84a, (b) Lemma 1.1.3] one finds that $[\mu_{h,v}]$ is actually an element of $\mathcal{M}_{\underline{\mathbf{G}}_{F_v}}(E_w) = \mathcal{M}_{\underline{\mathbf{G}}_{F_v}}(F_v)$, i.e. defined over $F_v$. In fact, the geometric conjugacy class $[\mu_{h,v}] \in \mathcal{M}_{\underline{\mathbf{G}}_{F_v}}(E_w)$ contains a cocharacter

$$\mathbb{G}_{m,\mathcal{O}_{E_w}} \to \underline{\mathbf{G}}_{\mathcal{O}_{E_w}},$$

defined over the valuation ring $\mathcal{O}_{E_w}$ (See [Kim18, Lemma 3.3.11]).

In summary, we get a pair $(\underline{\mathbf{G}}_{\mathcal{O}_{F_v}}, [\mu_{h,v}]) = (\mathbf{GL}_{n+1,\mathcal{O}_{F_v}} \times \mathbf{GL}_{n,\mathcal{O}_{F_v}}, [\mu_{h,v}])$, composed of a $F_v$-reductive group and a minuscule $\underline{\mathbf{G}}(F_v)$-conjugacy class $[\mu_{h,v}] \in \mathcal{M}_{\underline{\mathbf{G}}_{F_v}}(F_v)$. Moreover, if we make the standard choices[12]

$$\mathbf{T}_w := \{\text{diagonal matrices}\}$$

and

$$\mathbf{B}_w := \{\text{upper-triangular matrices}\} \subset \mathbf{GL}_{n+1,\mathcal{O}_{F_v}} \times \mathbf{GL}_{n,\mathcal{O}_{F_v}},$$

then the cocharacter $\mu_{h,v}$ given above, is the unique one in the class $[\mu_{h,v}]$ that is dominant with respect to $\mathbf{B}_w$.

Following §IV, we will attach to the pair $(\mathbf{GL}_{n+1,\mathcal{O}_{F_v}} \times \mathbf{GL}_{n,\mathcal{O}_{F_v}}, [\mu_{h,v}])$ the Hecke polynomial $\mathrm{H}_w(X)$ with coefficients in the Hecke algebra

$$\mathcal{H}(\mathbf{GL}_{n+1}(F_v) /\!\!/ \mathbf{GL}_{n+1}(\mathcal{O}_{F_v}), \mathbb{Z}[q_v^{\pm 1/2}]) \times \mathcal{H}(\mathbf{GL}_{n}(F_v) /\!\!/ \mathbf{GL}_{n}(\mathcal{O}_{F_v}), \mathbb{Z}[q_v^{\pm 1/2}]).$$

#### VII.2.3.2 Dual group

The complex dual of $\mathbf{GL}_{n+1,F_v} \times \mathbf{GL}_{n,F_v}$ is

$$\widehat{\mathbf{GL}}_{n+1,F_v} \times \widehat{\mathbf{GL}}_{n,F_v} = \mathbf{GL}_{n+1}(\mathbb{C}) \times \mathbf{GL}_{n}(\mathbb{C}).$$

Let $(\widehat{\mathbf{B}}_w, \widehat{\mathbf{T}}_w)$ be the standard Borel pair dual to $(\mathbf{B}_w, \mathbf{T}_w)$, that is the upper triangular matrices and its maximal torus of diagonal matrices. For convenience, we write $\widehat{\mathbf{B}}_w = \widehat{\mathbf{B}}_{v,1} \times \widehat{\mathbf{B}}_{v,2}$, $\widehat{\mathbf{T}}_w = \widehat{\mathbf{T}}_{v,1} \times \widehat{\mathbf{T}}_{v,2}$, and $W(\widehat{\mathbf{T}}_w) := W(\mathbf{GL}_{n+1}(\mathbb{C}) \times \mathbf{GL}_n(\mathbb{C}), \widehat{\mathbf{T}}_w)$ for the Weyl group. The Galois group $\mathrm{Gal}(F_v^{un}/F_v)$ acts trivially on dual of $\mathbf{GL}_{n+1,F_v} \times \mathbf{GL}_{n,F_v}$, since it a split group, i.e.

$${}^{L}(\mathbf{GL}_{n+1,F_v} \times \mathbf{GL}_{n,F_v}) = \mathbf{GL}_{n+1}(\mathbb{C}) \times \mathbf{GL}_{n}(\mathbb{C}) \times \mathrm{Gal}(F_v^{un}/F_v).$$

#### VII.2.3.3 The character $\widehat{\mu}_{h,v}$

Under the identification $X^*(\widehat{\mathbf{T}}_w) = X_*(\mathbf{T}_w)$, the Weyl orbit of $\mu_{h,v}$ corresponds to a Weyl $W(\widehat{\mathbf{T}}_w)$-orbit of characters of $\widehat{\mathbf{T}}_w$. There is a unique $\widehat{\mu}_{h,v} \in X^*(\widehat{\mathbf{T}}_w)$ in this Weyl orbit that is dominant with respect to the Borel subgroup $\widehat{\mathbf{B}}_w$, and it is explicitly given on $\mathbb{C}$-points by

$$\widehat{\mu}_{h,v} \colon \mathbf{T}_w(\mathbb{C}) \longrightarrow \mathbb{C}, \qquad (\mathrm{diag}(z_1, \cdots, z_{n+1}), \mathrm{diag}(z'_1, \cdots, z'_n)) \longmapsto z_1 z'_1.$$

[12] We use a subscript $w$ here, to keep track of the place $w$ aboce $v$ used to indentofy $\underline{G}_v$ with $\mathbf{GL}_{n+1,F_v} \times \mathbf{GL}_{n,F_v}$.

#### VII.2.3.4 The representation

The representation

$$r\colon\ \mathbf{GL}_{n+1}(\mathbb{C}) \times \mathbf{GL}_n(\mathbb{C}) \longrightarrow \mathbf{GL}_{n(n+1)},$$

we are interested in is the irreducible representation whose highest weight relative to the Borel pair $(\widehat{\mathbf{B}}_w, \widehat{\mathbf{T}}_w)$ is $\widehat{\mu}_{h,v}$ [BR94, §5.1]. Let $r_?$ be the standard ?-dimensional representation of $\mathbf{GL}_?(\mathbb{C})$, then $r$ is the representation on the 2-fold tensor product $r_{n+1} \otimes r_n$ defined as follows, for any $g = (g_1, g_2) \in \mathbf{GL}_{n+1}(\mathbb{C}) \times \mathbf{GL}_n(\mathbb{C})$,

$$r(g) = r_{n+1}(g_1) \otimes r_n(g_2).$$

Finally, extend $r$ to a representation of ${}^L(\mathbf{GL}_{n+1,F_v} \times \mathbf{GL}_{n,F_v})$ (also called $r$) by letting the Galois group act trivially everywhere.

#### VII.2.3.5 The Hecke polynomial

**VII.2.3.5.1 Definition.** Following Definition IV.3.0.1, we attach to the pair $(\underline{\mathbf{G}}_{F_v}, [\mu_{h,v}])$ the polynomial

$$H_w(z) = \det\left(z - q_v^{\langle \mu_{h,v}, \rho_v\rangle} r(g)\right) \in \mathbb{C}[\mathbf{GL}_{n+1}(\mathbb{C}) \times \mathbf{GL}_n(\mathbb{C})][z],$$

where, $\rho_v$ is the halfsum of all positive roots of $(\mathbf{B}_w, \mathbf{T}_w)$, thus $\langle \mu_{h,v}, \rho_v\rangle = 2n - 1$. The ring of coefficients of this polynomial, is the ring of class functions on $\mathbf{GL}_{n+1}(\mathbb{C}) \times \mathbf{GL}_n(\mathbb{C})$. A class function on $\mathbf{GL}_{n+1}(\mathbb{C}) \times \mathbf{GL}_n(\mathbb{C})$ restricts to a Weyl-invariant function on $\widehat{\mathbf{T}}_w$. Conversely, by a classical argument of Chevalley, the subalgebra of Weyl-invariants functions on $\widehat{\mathbf{T}}_w$ consists precisely of functions which arise from class functions on $\mathbf{GL}_{n+1}(\mathbb{C}) \times \mathbf{GL}_n(\mathbb{C})$. Therefore, one can identify $H_w(z)$ with a polynomial in $\mathbb{C}[\widehat{\mathbf{T}}_w]^{W(\widehat{\mathbf{T}}_w)}[t]$, which, abusing notation, we continue to denote by $H_w$:

$$H_w(t) = \det\left(z - q_v^{\langle \mu_{h,v}, \rho_v\rangle} r_{|\widehat{\mathbf{T}}_w}(t)\right) \in \mathbb{C}[\widehat{\mathbf{T}}_w][t].$$

Let $t = (\operatorname{diag}(x_1, \cdots, x_{n+1}), \operatorname{diag}(y_1, \cdots, y_n)) \in \widehat{\mathbf{T}}_w$, the polynomial then identifies with

$$H_w(t) = \prod_{i=1}^{n+1}\prod_{j=1}^{n}\left(z - q_v^{2n-1} x_i y_j\right) \in \mathbb{C}[\widehat{\mathbf{T}}_w][t].$$

Here, $x_i y_j$ denotes the function $x_i y_j\colon \widehat{\mathbf{T}}_w \to \mathbb{C}$ defined by

$$(\operatorname{diag}(x_1, \cdots, x_{n+1}), \operatorname{diag}(y_1, \cdots, y_n)) \mapsto x_i y_j.$$

Now, using the identification[13] $\mathbb{C}[\widehat{\mathbf{T}}_w] \simeq \mathcal{H}_{\mathbb{C}}(\mathbf{T}_w(F_v) /\!\!/ \mathbf{T}_w(\mathcal{O}_{F_v}))$, this function corresponds to the element $\mathbf{1}_{(g_i,h_j)T_c} \in \mathcal{H}_{\mathbb{C}}(\mathbf{T}_w(F_v) /\!\!/ \mathbf{T}_w(\mathcal{O}_{F_v}))$ where

$$g_i = \mathrm{diag}\,(\underbrace{1,\dots 1}_{i-1}, \varpi_v, 1\dots, 1) \text{ and } h_j = \mathrm{diag}(\underbrace{1,\dots 1}_{j-1}, \varpi_v, 1, \dots, 1).$$

Since the Weyl group $W(\widehat{\mathbf{T}}_w)$ permutes the $x_i$'s and $y_j$'s, it is clear that $H_w(t) \in \mathbb{C}[\widehat{\mathbf{T}}_w]^{W(\widehat{\mathbf{T}}_w)}[z]$. Subsequently, using the Satake isomorphism (Theorem III.10.0.1)

$$\mathbb{C}[\widehat{\mathbf{T}}_w]^{W(\widehat{\mathbf{T}}_w)} \simeq \mathcal{H}_{\mathbb{C}}(\mathbf{T}_w(F_v) /\!\!/ \mathbf{T}_w(\mathcal{O}_{F_v}))^{W(\mathbf{T}_w)} \simeq \mathcal{H}_{\mathbb{C}}(\underline{\mathbf{G}}(F_v) /\!\!/ \underline{\mathbf{G}}(\mathcal{O}_{F_v})),$$

we may also view $H_w$ as a polynomial with coefficients in the local spherical Hecke algebra.

**VII.2.3.5.2 Explicit polynomial.** In the remaining part of this section, we will give an explicit formulation for the Hecke polynomial with coefficients in the spherical local Hecke algebra, i.e. we will invert the Satake isomorphism. Let us begin by rewriting this polynomial in a more suitable form:

$$\begin{aligned} H_w(t) &= \prod_{j=1}^{n}\prod_{i=1}^{n+1}\left(z - q_v^{2n-1}x_i y_j\right) \\ &= \prod_{j=1}^{n}\left(\sum_{k=0}^{n+1}(-1)^k q_v^{(2n-1)k} X_k y_j^k z^{n+1-k}\right) \\ &= \sum_{k=0}^{n(n+1)}\Big(\sum_{(a_i)\in p_n(k)}\prod_{i=1}^{n}(-1)^{a_i}q_v^{2n-1a_i}X_{a_i}y_i^{a_i}\Big)z^{n(n+1)-k} \end{aligned}$$

where, $p_n(k) := \{(a_i)_{1\le i\le n} \in \mathbb{N}\colon \sum_{i=1}^{n} a_i = k, 0 \le a_i \le n+1\}$ and $X_?$ is the symmetric monomial associated to the monome $x_1x_2\cdots x_?$, for $1 \le ? \le n+1$. Here, the symmetric permutation group $S_n$ acts on the set of partitions $p_n(j)$, and yields

$$\begin{aligned} H_w(t) &= \sum_{k=0}^{n(n+1)}\Big(\sum_{(a_i)\in p_n(k)}\prod_{i=1}^{n}(-1)^{a_i}q_v^{(2n-1)a_i}X_{a_i}y_i^{a_i}\Big)z^{n(n+1)-k} \\ &= \sum_{k=0}^{n(n+1)}\Big(\sum_{(a_i)\in p_n(k)/S_n}(-1)^{\sum a_i}q_v^{2n-1\sum_{i=1}^{n}a_i}\prod_{i=1}^{n}X_{a_i}\cdot\prod_{i=1}^{n}\big(\sum_{j=1}^{n}y_j^{a_i}\big)\Big)z^{n(n+1)-k} \\ &= \sum_{k=0}^{n(n+1)}(-1)^k q_v^{k(2n-1)}\Big(\sum_{(a_i)\in p_n(k)/S_n}\prod_{i=1}^{n}X_{a_i}\cdot\prod_{i=1}^{n}\big(\sum_{j=1}^{n}y_j^{a_i}\big)\Big)z^{n(n+1)-k} \\ &= \sum_{k=0}^{n(n+1)}(-1)^k q_v^{k(2n-1)}\Big(\sum_{(a_i)\in p_n(k)/S_n}\prod_{i=1}^{n}X_{a_i}\cdot\prod_{i=1}^{n}Y^{(a_i)}\Big)z^{n(n+1)-k} \end{aligned}$$

[13] This identification is obtained as follows: On one hand, by definition of the dual torus, the ring of algebraic functions on $\widehat{\mathbf{T}}_w$ are precisely the elements of $\mathbb{C}[X_*(\mathbf{T}_w)]$. On the other hand, sending any $F_v$-algebraic cocharacter $\chi\colon \mathbb{G}_{m,F_v} \to \mathbf{T}_w$ to $1_{\chi(\varpi_v)\mathbf{T}_w(\mathcal{O}_{F_v})}$, yields an identification $\mathbb{C}[X_*(\mathbf{T}_w)] \simeq \mathcal{H}_{\mathbb{C}}(\mathbf{T}_w(F_v) /\!\!/ \mathbf{T}_w(\mathcal{O}_{F_v}))$.

where, $Y^{(?)}$ denotes the power sum symmetric monomial $\sum_{i=1}^{n} y_i^?$, for $? \geq 1$. Set $Y_?$ for the symmetric monomial associated to the monome $y_1 y_2 \cdots y_?$, for $1 \leq ? \leq n$. For any $k \geq 1$, the Newton–Girard formula says:

$$Y^{(k)} = \begin{cases} (-1)^k k Y_k + \sum_{i=1}^{k-1} (-1)^{k-1+i} Y_{k-i} Y^{(i)} & \text{if } 1 \leq k \leq n, \\ \sum_{i=k-n}^{k-1} (-1)^{k-1+i} Y_{k-i} Y^{(i)} & \text{if } n < k. \end{cases}$$

There exists then, a polynomial in $n$ variables $Q_k(z_1, \cdots, z_n) \in \mathbb{Z}[z_1, \cdots, z_n]$ such that the power sum $Y^{(k)}$ is given by

$$Y^{(k)} = Q_k(Y_1, \ldots, Y_n).$$

Set $T_{k,\star}$ for the Hecke operator $\mathbf{GL}_\star(\mathcal{O}_{F_v}) \cdot \mathrm{diag}(\underbrace{\varpi_v, \cdots, \varpi_v}_{k}, 1, \cdots, 1) \cdot \mathbf{GL}_\star(\mathcal{O}_{F_v})$, for $\star \in \{V, W\}$. The Satake isomorphism $\mathbb{C}[\widehat{\mathbf{T}}_w]^{W(\widehat{\mathbf{T}}_w)} \simeq \mathcal{H}_{\mathbb{C}}(\underline{\mathbf{G}}(F_v) /\!\!/ \underline{\mathbf{G}}(\mathcal{O}_{F_v}))$ yields the following identifications

$$\begin{aligned} X_k, (1 \leq k \leq n+1) &\longleftrightarrow q_v^{-\frac{(n+1-k)k}{2}} T_{k,V}, \\ Y_k, (1 \leq k \leq n) &\longleftrightarrow q_v^{-\frac{(n-k)k}{2}} T_{k,W} \end{aligned} \tag{VII.4}$$

Indeed, for any fixed integer $? \geq 1$ and every integer $1 \leq i \leq ?$ we consider the following minuscule[14] cocharcater $\lambda_i \colon \mathbb{G}_{m,F} \to \mathbf{GL}_?$ given on $F_v$ points by

$$\lambda_i \colon F_v^\times \longrightarrow \mathbf{GL}_?(F), \qquad t \mapsto \mathrm{diag}(\underbrace{t, \cdots, t}_{i-\text{tuple}}, 1 \cdots, 1).$$

The above identification is an immediate application of Proposition IV.4.0.1, where we also use the fact that the Modulus function for $\mathbf{GL}_\star$ with respect to the Borel subgroup of upper triangular matirces $B_?$ is given by $\delta_{B_?}(\mathrm{diag}(\varpi^{a_i})) = q_v^{-\sum_{j=1}^{?}(?+1-2j)a_j}$, for any diagonal matrix $\mathrm{diag}(\varpi^{a_i}) \in \mathbf{GL}_?(F_v)$.

Consequently we can now write the Hecke polynomial with coefficients in the spherical local Hecke algebra:

$$\begin{aligned} H_w(t) &= \sum_{j=0}^{n(n+1)} (-1)^k q_v^{k(2n-1)} \Big( \sum_{(a_i) \in p_n(k)/S_n} \prod_{i=1}^{n} q_v^{-\frac{(n+1-a_i)a_i}{2}} T_{a_i,V} \otimes \prod_{i=1}^{n} T_W^{(a_i)} \Big) z^{n(n+1)-k} \\ &= \sum_{j=0}^{n(n+1)} (-1)^k q_v^{\frac{3k(n-1)+\sum_{i=1}^{n} a_i^2}{2}} \Big( \sum_{(a_i) \in p_n(k)/S_n} \prod_{i=1}^{n} T_{a_i,V} \otimes \prod_{i=1}^{n} T_W^{(a_i)} \Big) z^{n(n+1)-k} \end{aligned}$$

where, $T_W^{(a_i)} := Q_{a_i}\left( q_v^{-\frac{n-1}{2}} T_{1,W}, \ldots, q_v^{-\frac{(n-j)j}{2}} T_{j,W}, \ldots, T_{n,W} \right)$.

**Example** VII.2.3.1 (Case $n = 1$). *we obtain in this situation the following polynomial*

$$H_w(t) = t^2 - q_v^{1/2}(x_1 + x_2) y_1 t + q_v x_1 x_2 y_1^2 \in \mathbb{C}[\widehat{\mathbf{T}}_w][t],$$

[14] i.e. the representation $\mathrm{Ad} \circ \lambda_i$ of $\mathbb{G}_m$ on $\mathrm{Lie}(\mathbf{GL}_?)$ has no weights other than $1, 0, -1$.

*which corresponds to the Hecke polynomial*

$$H_w(z) = z^2 - (T_{1,V} \otimes T_{1,W})\, z + q_v\, (T_{2,V} \otimes T_{1,W}^2),$$

*where* $T_{1,V}, T_{2,V} \in \mathcal{C}_c(\mathbf{GL}_2(F_v) /\!\!/ \mathbf{GL}_2(\mathcal{O}_{F_v}), \mathbb{Z})$ *are the Hecke operators corresponding to*

$$\mathbf{GL}_2(\mathcal{O}_{F_v})\, \mathrm{diag}(\varpi_v, 1)\, \mathbf{GL}_2(\mathcal{O}_{F_v}), \mathrm{diag}(\varpi_v, \varpi_v)\, \mathbf{GL}_2(\mathcal{O}_{F_v})$$

*respectively, and* $T_{1,W}^? \in \mathcal{C}_c(F_v^\times / \mathcal{O}_{F_v}^\times, \mathbb{Z})$ *is the Hecke operator corresponding to* $\varpi_v^? \mathcal{O}_{F_v}^\times$.

**Example** VII.2.3.2 (Case $n = 2$). *In this case, we have*

$$\begin{aligned} H_w(t) &= \prod_{i=1}^{2} (z^3 - q_v^{\frac{3}{2}} X_1 y_i z^2 + q_v^3 X_2 y_i^2 z - q_v^{\frac{9}{2}} X_3 y_i^3) \\ &= z^6 - q_v^{\frac{3}{2}} X_1 (y_1 + y_2) z^5 + q_v^3 \big(X_2 (y_1^2 + y_2^2) + X_1^2 y_1 y_2\big) z^4 \\ &- q_v^{\frac{9}{2}} \big(X_1 X_2 y_1 y_2 (y_1 + y_2) + X_3 (y_1^3 + y_2^3)\big) z^3 \\ &+ q_v^6 \big(X_1 X_3 y_1 y_2 (y_1^2 + y_2^2) + X_2^2 (y_1 y_2)^2\big) z^2 \\ &- q_v^{\frac{15}{2}} \big(X_2 X_3 (y_1 y_2)^2 (y_1 + y_2)\big) z + q_v^9 X_3^2 (y_1 y_2)^3 \end{aligned}$$

*Using (VII.4) we also get the identifications*

$$y_1^2 + y_2^2 = (y_1 + y_2)^2 - 2 y_1 y_2 \longleftrightarrow q_v^{-\frac{1}{2}} T_{1,W}^2 - 2 T_{2,W}$$

$$y_1^3 + y_2^3 = (y_1 + y_2)((y_1 + y_2)^2 - 3 y_1 y_2) \longleftrightarrow q_v^{-\frac{1}{2}} T_{1,W} (q_v^{-1} T_{1,W}^2 - 3 T_{2,W}).$$

*Therefore the Hecke polynomial is given by*

$$\begin{aligned} H_w(t) = z^6 &- T_{1,V} \otimes T_{1,W} z^5 + q \big(T_{2,V} \otimes (T_{1,W}^2 - 2 q_v T_{2,W}) + T_{1,V}^2 \otimes T_{2,W}\big) z^4 \\ &- q_v^2 (1 \otimes T_{1,W}) \big(T_{1,V} T_{2,V} \otimes T_{2,W} + q_v T_{3,V} \otimes (T_{1,W}^2 - 3 q_v T_{2,W})\big) z^3 \\ &+ q_v^4 (1 \otimes T_{2,W}) \big(T_{1,V} T_{3,V} \otimes T_{1,W}^2 - 2 q_v^2 + T_{2,V}^2 \otimes T_{2,W}\big) z^2 \\ &- q_v^6 \big(T_{2,V} T_{3,V} \otimes T_{1,W} T_{2,W}^2\big) z \\ &+ q_v^9 T_{3,V}^2 \otimes T_{2,W}^3. \end{aligned}$$

### VII.2.4 Split local setting

*To ease the reading, we switch to a local notation and omit all subscripts $v$ and $w$.*

Using the fixed basis for $V$ (compatible with $W$), we get to the following situation

$\mathbf{H} = \Delta\,\mathbf{GL}_n \hookrightarrow \mathbf{G} = \mathbf{GL}_{n+1} \times \mathbf{GL}_n$, where the embedding on the first factor is given by

$$\iota\colon \begin{pmatrix} \boxed{\;*\;} \end{pmatrix} \mapsto \begin{pmatrix} \boxed{\;*\;} & \\ & 1 \end{pmatrix}$$

We would like to describe the quotient $H\backslash G/K$ where $K = \mathbf{G}(\mathcal{O}_F)$. Set $K_1 = \mathbf{GL}_{n+1}(\mathcal{O}_F)$ and $K_2 = \mathbf{GL}_n(\mathcal{O}_F)$. For every $(g_1, g_2) \in G$, we have $H(g_1, g_2) = H(\iota(g_2)^{-1} g_1, 1) \in H\backslash G$. This induces a bijection $H\backslash G \simeq \mathbf{GL}_{n+1}(F)$, given by $H(g_1, g_2) \mapsto \iota(g_2)^{-1} g_1$. The natural right action of $G$ on the quotient space $H\backslash G$ corresponds to the following right action on $\mathbf{GL}_{n+1}(F)$:

$$\forall (g_1, g_2) \in G,\ \forall g \in \mathbf{GL}_{n+1}(F), \quad g \cdot (g_1, g_2) = \iota(g_2)^{-1} g\, g_1 \in \mathbf{GL}_{n+1}(F).$$

This isomorphism of right $G$-spaces, induces the following bijection

$$\begin{aligned} H\backslash G/K &\simeq \{g \cdot K \colon g \in \mathbf{GL}_{n+1}(F)\} \\ &\simeq \{\iota(K_2)^{-1} g\, K_1 \colon a \in \mathbf{GL}_{n+1}(F), \} \\ &\simeq \iota(K_2)\backslash\, \mathbf{GL}_{n+1}(F)/K_1. \end{aligned}$$

Let $\mathbf{B}_1 \subset \mathbf{GL}_{n+1}$ (resp. $\mathbf{B}_2 \subset \mathbf{GL}_n$) be the Borel subgroup of upper triangular matrices. We denote their respective opposite Borel subgroups by $\overline{\mathbf{B}}_1$ and $\overline{\mathbf{B}}_2$. Consider the Borel subgroup $\mathbf{B}^{\sim} := \mathbf{B}_1 \times \overline{\mathbf{B}}_2 \subset \mathbf{G}$. Let $\mathbf{U}_1$ (resp $\mathbf{U}_2$) be the unipotent radical of $\mathbf{B}_1$ (resp. $\mathbf{B}_2$) and $\mathbf{T} = \mathbf{T}_1 \times \mathbf{T}_2 \subset \mathbf{B}^{\sim}$ be the split maximal torus where $\mathbf{T}_1$ (resp. $\mathbf{T}_2$) is the split maximal diagonal torus of $\mathbf{GL}_{n+1}$ (resp. $\mathbf{GL}_n$). The set of $\mathbf{B}^{\sim}$-antidominant diagonals is $T^{\sim} = T_1^- \times T_2^+$, where

$$\begin{aligned} T_1^- &= \{\mathrm{diag}(\varpi^{a_k})_{1\le k\le n+1}, \colon a_i \in \mathbb{Z} \text{ such that } a_1 \ge \cdots \ge a_{n+1}\} \cdot \mathbf{T}_1(\mathcal{O}_F), \\ T_2^+ &= \left\{\mathrm{diag}(\varpi^{b_k})_{1\le k\le n}, \colon b_i \in \mathbb{Z} \text{ such that } b_1 \le \cdots \le b_n\right\} \cdot \mathbf{T}_2(\mathcal{O}_F). \end{aligned}$$

**Proposition** VII.2.4.1. *Every class in $H\backslash G/K$ admits a representative of the form*

$$\left( \begin{pmatrix} \varpi^{a_1} & & & 1 \\ & \ddots & & \vdots \\ & & \varpi^{a_n} & 1 \\ & & & \varpi^{c} \end{pmatrix}, \begin{pmatrix} \varpi^{b_1} & & \\ & \ddots & \\ & & \varpi^{b_n} \end{pmatrix} \right) \in G,$$

*for some $c \in \mathbb{Z}$ and $\left(\mathrm{diag}(\varpi^{a_k})_{1\le k\le n+1}, \mathrm{diag}(\varpi^{b_k})_{1\le k\le n}\right) \in T^{\sim}$ (with $a_{n+1} := 0$).*

*Proof.* Let us first prove that the set

$$\left\{ \varpi^c \begin{pmatrix} \varpi^{a_1} & & & \varpi^{b_1} \\ & \ddots & & \vdots \\ & & \varpi^{a_n} & \varpi^{b_n} \\ & & & 1 \end{pmatrix} : c \in \mathbb{Z}, a_n \geq b_n \text{ and } k < l \Rightarrow a_k - a_l \geq b_k - b_l \geq 0 \right\},$$

is a set of class representatives for the quotient

$$\iota(K_2) \backslash \mathbf{GL}_{n+1}(F) / K_1.$$

Observe that $\iota(\mathbf{GL}_n) \times \chi_{1,n+1}(\mathbb{G}_m)$ is a Levi factor of the parabolic subgroup:

$$\mathbf{P} := \begin{pmatrix} \boxed{\mathbf{GL}_n} & \begin{matrix} * \\ \vdots \\ * \end{matrix} \\ & \mathbb{G}_m \end{pmatrix} = \begin{pmatrix} \boxed{\mathbf{GL}_n} & \\ & \mathbb{G}_m \end{pmatrix} \ltimes \begin{pmatrix} 1 & & & * \\ & \ddots & & \vdots \\ & & \ddots & * \\ & & & 1 \end{pmatrix},$$

where, the second factor of the Levi decomposition above is the unipotent radical $\mathbf{U_P}$ of $\mathbf{P}$. Therefore, every class in $\iota(K_2) \backslash \mathbf{GL}_{n+1}(F)/K_1$ has a representatives of the form $\varpi^c \iota(g) u$ for some $c \in \mathbb{Z}$, $g \in \mathbf{GL}_n(F)$ and $u \in \mathbf{U_P}(F)$. By the Cartan decomposition for $\mathbf{GL}_n(F)$, we know there exist $k, k' \in K_2$ such that $k' g k = \operatorname{diag}(\varpi^{a_k})_{1 \leq k \leq n}$ with $a_1 \geq \cdots \geq a_n$. Hence

$$\iota(K_2) \varpi^c g u K_1 = \iota(K_2) \varpi^c \iota(k' g k) \iota(k)^{-1} u \, \iota(k) K_1.$$

Since $\iota(k)^{-1} u \, \iota(k) \in \mathbf{U_P}(F)$, then each class in $\iota(K_2) \backslash \mathbf{GL}_{n+1}(F)/K_1$ has a representative of the form

$$\varpi^c \begin{pmatrix} \varpi^{a_1} & & & \varpi^{b_1} d_1 \\ & \ddots & & \vdots \\ & & \varpi^{a_n} & \varpi^{b_n} d_n \\ & & & 1 \end{pmatrix}$$

For some $c, a_k, b_k \in \mathbb{Z}$ such that $k < l \Rightarrow a_k - a_l \geq 0$ and, $d_k \in \mathcal{O}_F$.

- $\underline{d_k \neq 0}$ : Suppose some of the $d_k$'s is zero. Using a unipotent matrix from the right we can replace $\mathrm{Column}_{n+1}$ by $\mathrm{Column}_{n+1} + \mathrm{Column}_k$. Thus we get a new matrix with $d_k = 1$ and $b_k$ is equal to $a_k$.

- $\underline{d_k = 1}$ : Suppose then that all $d_k$'s are non zero. Conjugation by the matrix

$\mathrm{diag}(d_1, \ldots, d_n, 1) \in \iota(K_2)$ shows that one can ignore the $d_k$'s:

$$\varpi^c \begin{pmatrix} \varpi^{a_1} & & & \varpi^{b_1} \\ & \ddots & & \vdots \\ & & \varpi^{a_n} & \varpi^{b_n} \\ & & & 1 \end{pmatrix}$$

- $\underline{a_n \geq b_n}$ : If we have $a_n < b_n$, we may change $b_n$ to $a_n$ by replacing the $\mathrm{Column}_{n+1}$ by $\mathrm{Column}_{n+1} + \mathrm{Column}_n$.

- (*) $\underline{b_1 \geq \cdots \geq b_n}$ : Let $k < l$ and suppose that $b_k < b_l$. We can find a matrix in $\iota(K_2)$ taking $\mathrm{Row}_l$ to $\mathrm{Row}_l + \mathrm{Row}_k$, thus we get in the $k^{th}$ row $\varpi^{b_k} + \varpi^{b_l} = \varpi^{b_k}(1 + \varpi^{b_l - b_k})$. We kill the invertible element $1 + \varpi^{b_k - b_l} \in \mathcal{O}_F^\times$ by adjoint action of a diagonal matrix in $\iota(K_2)$ having $(1 + \varpi^{b_l - b_k})^{-1}$ in the $l^{th}$ component. Now we kill the element that appears in the $(l, k)$ position of the new matrix: use the action of $K_1$ on columns to take $\mathrm{Column}_k$ to $\mathrm{Column}_k - \varpi^{a_k - a_l}(1 + \varpi^{b_l - b_k})^{-1}\mathrm{Column}_l$. We thus have changed the matrix by replacing -only- the old $b_l$ by $b_k$.

- $\underline{a_k - a_l \geq b_k - b_l}$ : Fix a $k \leq n$. Suppose that for some $l > k$ we have $c_{k,l} := (b_k - b_l) - (a_k - a_l) > 0$. Let $l'$ be such that $c_{k,l'} = \max_{l \geq k} c_{k,l}$. Take $\mathrm{Row}_k$ to

  $$\mathrm{Row}_k + \varpi^{a_k - a_{l'}}\mathrm{Row}_{l'} = \mathrm{Row}_k + \varpi^{b_k - b_{l'} - c_{k,l'}}\mathrm{Row}_{l'}, \quad (a_k - a_{l'} \geq 0).$$

  Note that for every $l \geq k$ we have

  $$(b_k - c_{k,l'} - b_l) - (a_k - a_l) \leq 0.$$

  We clear the component appearing in the $(k, l)$ position by replacing $\mathrm{Column}_{l'}$ by $\mathrm{Column}_{l'} - \mathrm{Column}_k$. Therefore, we may[15] replace in the old matrix (only) $b_k$ by $b_k - c_{k,l'}$. By doing so, we do not alter the previous required inequalities.

  Suppose there is an $l \geq k$ such that the new $b_k$ is $< b_l$. Using the step (*) above, we saw that we can replace $b_l$ by $b_k$. By doing so, we do not alter the required inequality $(b_k - b_l) - (a_k - a_l) \leq -(a_k - a_l) \leq 0$.

[15]The last coefficient of the $k$-th row is $\varpi_k^b + \varpi^{b_{l'} + a_k - a_{l'}}$ which is not of the form $\varpi^{b'_k}$. Writing this as $\varpi^{b_{l'} + a_k - a_{l'}}u$ with $u$ invertible, one can then use yet another conjugation to get it to $\varpi^{b'_k}$ with $b'_k = b_{l'} + a_k - a_{l'} = b_k - c_{k,l'}$.

Now, every class $\iota(K_2)\varpi^c \begin{pmatrix} \varpi^{a_1} & & & \varpi^{b_1} \\ & \ddots & & \vdots \\ & & \varpi^{a_n} & \varpi^{b_n} \\ & & & 1 \end{pmatrix} K_1$ corresponds to

$$H\left(\begin{pmatrix} \varpi^{a_1-b_1} & & & 1 \\ & \ddots & & \vdots \\ & & \varpi^{a_n-b_n} & 1 \\ & & & \varpi^c \end{pmatrix}, \begin{pmatrix} \varpi^{-b_1-c} & & \\ & \ddots & \\ & & \varpi^{-b_n-c} \end{pmatrix}\right) K \in H\backslash G/K,$$

such that $-b_1 \leq \cdots \leq -b_n$ and $a_1 - b_1 \geq \cdots \geq a_n - b_n \geq 0$. This completes the proof of the proposition. □

**Lemma** VII.2.4.1. *For $\star \in \{1,2\}$, let*

$$g_\star = \left(\begin{pmatrix} \varpi^{a_{1,\star}} & & & 1 \\ & \ddots & & \vdots \\ & & \varpi^{a_{n,\star}} & 1 \\ & & & \varpi^{c_\star} \end{pmatrix}, \begin{pmatrix} \varpi^{b_{1,\star}} & & \\ & \ddots & \\ & & \varpi^{b_{n,\star}} \end{pmatrix}\right),$$

*for some $a_{k,\star}, b_{k,\star}, c_\star \in \mathbb{Z}$ such that for $\star \in \{1,2\}$ the sequences $(a_{k,\star})_k$ is non-increasing and $(b_{k,\star})_k$ is non-decreasing. If $Hg_1K = Hg_2K$ then $c_1 = c_2$ and $a_{k,1} - b_{k,1} = a_{k,2} - b_{k,2}$ for each $1 \leq k \leq n$.*

*Proof.* Suppose that the classes of $g_1$ and $g_2$ are the same in $H\backslash G/K$. This is true if and only if there exists $k_2 \in K_2$ and $k_1 \in K_1$ such that

$$\iota(k_2)^{-1}\begin{pmatrix} \varpi^{a_{1,1}-b_{1,1}} & & & \varpi^{-b_{1,1}} \\ & \ddots & & \vdots \\ & & \varpi^{a_{n,1}-b_{n,1}} & \varpi^{-b_{n,1}} \\ & & & \varpi^{c_1} \end{pmatrix} k_1 = \begin{pmatrix} \varpi^{a_{1,2}-b_{1,2}} & & & \varpi^{-b_{1,2}} \\ & \ddots & & \vdots \\ & & \varpi^{a_{n,2}-b_{n,2}} & \varpi^{-b_{n,2}} \\ & & & \varpi^{c_2} \end{pmatrix}$$

It is clear that $k_1 \in K_1 \cap \mathbf{P}(F)$, i.e.

$$k_1 = \begin{pmatrix} \boxed{k_1'} & \begin{matrix} u_1 \\ \vdots \\ u_n \end{matrix} \\ & u_{n+1} \end{pmatrix}.$$

Here, $k_1'$ belongs to $K_2$ and $u_{n+1} \in \mathcal{O}_F^\times$. The equality above implies $\varpi^{c_2} = u_{n+1}\varpi^{c_1}$ (i.e.,

$c_1 = c_2$) and

$$k_2 \begin{pmatrix} \varpi^{a_{1,1}-b_{1,1}} & & \\ & \ddots & \\ & & \varpi^{a_{n,1}-b_{n,1}} \end{pmatrix} k_1' = \begin{pmatrix} \varpi^{a_{1,2}-b_{1,2}} & & \\ & \ddots & \\ & & \varpi^{a_{n,2}-b_{n,2}} \end{pmatrix}$$

Since both diagonal matrices are $\overline{\mathbf{B}}_2$-antidominant and represent the same class in $K_2\backslash \mathbf{GL}_n(F)/K_2$ they must be equal. $\square$

**Lemma** VII.2.4.2. *For any integers $c, a_1, \ldots, a_n, b_1, \ldots b_n$, consider the matrix*

$$g = \left( \varpi^c \begin{pmatrix} \varpi^{a_1} & & & 1 \\ & \ddots & & \vdots \\ & & \varpi^{a_n} & 1 \\ & & & 1 \end{pmatrix}, \begin{pmatrix} \varpi^{b_1} & & \\ & \ddots & \\ & & \varpi^{b_n} \end{pmatrix} \right) \in G.$$

*The stabilizer* $\mathrm{Stab}_H(gK)$ *is equal to*

$$\Big\{\Delta((h_{ij})) \in H \colon h_{ii} \in \mathcal{O}_F, h_{ij} \in \varpi^{\max\{a_i-a_j, b_i-b_j\}}\mathcal{O}_F \forall i \neq j,$$
$$\sum_{j=1}^{n} h_{ij} \in 1 + \varpi^{a_i}\mathcal{O}_F, 1 \leq \forall i \leq n\Big\}.$$

*Proof.* Set $\mathrm{diag}(a) = \mathrm{diag}(\varpi^{a_1}, \ldots, \varpi^{a_n}, 1)$ and similarly $\mathrm{diag}(b) = \mathrm{diag}(\varpi^{b_1}, \ldots, \varpi^{b_n})$. A matrix $h = (h_{ij}) \in \mathbf{GL}_n(F)$ verifies $\Delta(h) \in \mathrm{Stab}_H(gK)$ by definition if and only if

$$\begin{cases} \mathrm{diag}(-a)u_0^{-1}\iota(h)u_0\,\mathrm{diag}(a)K_1 & = K_1 \\ \mathrm{diag}(-b)h\,\mathrm{diag}(b)K_2 & = K_2, \end{cases}$$

which is equivalent to

$$\begin{pmatrix} \boxed{\mathrm{diag}(-a)h\,\mathrm{diag}(a)} & \begin{matrix} \varpi^{-a_1}(-1+\sum_{j=1}^n h_{1j}) \\ \vdots \\ \varpi^{-a_n}(-1+\sum_{j=1}^n h_{nj}) \end{matrix} \\ & 1 \end{pmatrix} \in K_1 \text{ and } \mathrm{diag}(-b)h\,\mathrm{diag}(b) \in K_2.$$

Therefore $h \in \mathrm{diag}(a)K_2\,\mathrm{diag}(-a) \cap \mathrm{diag}(b)K_2\,\mathrm{diag}(-b)$ and for all $1 \leq i \leq n$ we must have $\sum_{j=1}^n h_{ij} \in 1 + \varpi^{a_i}\mathcal{O}_F$. This proves that $\mathrm{Stab}_H(gK)$ is equal to

$$\{\Delta((h_{ij})) \in H \colon h_{ii} \in \mathcal{O}_F,\ h_{ij} \in \varpi^{c_{ij}}\mathcal{O}_F \text{ and } \sum_{j=1}^{n} h_{ij} \in 1 + \varpi^{a_i}\mathcal{O}_F, 1 \leq \forall i \leq n\}.$$

where, $c_{ij} := \max\{a_i - a_j, b_i - b_j\}$ for $i \neq j$. $\square$

In the following proposition we compute the determinant of $H$-stabilizers of cosets in $G/K$.

**Proposition** VII.2.4.2. *Let $c, a_1, \ldots, a_n, b_1, \ldots b_n$ be any integers, and consider the element*

$$g = \left( \varpi^c \begin{pmatrix} \varpi^{a_1} & & & 1 \\ & \ddots & & \vdots \\ & & \varpi^{a_n} & 1 \\ & & & 1 \end{pmatrix}, \begin{pmatrix} \varpi^{b_1} & & \\ & \ddots & \\ & & \varpi^{b_n} \end{pmatrix} \right) \in G.$$

*Set $c_{ij} := \max\{a_i - a_j, b_i - b_j\}$ if $i \neq j$. We then have* $\det(\mathrm{Stab}_H(gK)) =$

$$\begin{cases} \mathcal{O}_F^\times, & \text{if there exists } i \text{ such that } a_i \leq 0, \\ \mathcal{O}_F^\times, & \text{if there exists a pair } (i,j) \text{ such that } i \neq j \text{ and } c_{ij} \leq 0, \\ 1 + \varpi^{\min(\{a_i : 1 \leq i \leq n\} \cup \{c_{ij} : 1 \leq i \neq j \leq n\})} \mathcal{O}_F & \text{if } c_{i,j} > 0 \text{ for all } i \neq j \text{ and } a_i > 0. \end{cases}$$

*Proof.* For the first two cases, for each $t \in \mathcal{O}_F^\times$ one can give explicitly an element $g_t \in H$ stabilizing $gK$ with determinant $t$:

- Let $i$ be an index for which $a_i \leq 0$. One can take $g_t = \mathrm{diag}(1, \ldots, 1, t, 1, \ldots, 1) \in K_2$, with $t$ in the $i^{th}$ position.
- Let $i$ and $j$ be indices, for which $c_{ij} \leq 0$. One can take $g_t$ to be the matrix with $t$ in the $i^{th}$ diagonal component, 1's elsewhere in the diagonal, $1 - t$ in the $ij$ component and zeros everywhere else.

Suppose now, that for all $1 \leq i, j \leq n$ we have $c_{ij} > 0$ and, $a_i > 0$. In this case a matrix $h = (h_{ij})$ belongs to $\mathrm{Stab}_H(gK)$ only if

$$h_{ij} \in \varpi^{c_{ij}} \mathcal{O}_F, \forall 1 \leq i \neq j \leq n \text{ and } h_{ii} \in 1 + \varpi^{\min\{a_i, c_{ij} \colon 1 \leq j \leq n, j \neq i\}} \mathcal{O}_F, \forall i \leq n.$$

Therefore, for all $h = (h_{ij}) \in \mathrm{Stab}_H(gK)$ we have $\det(h) \in 1 + \varpi^{\min\{a_i, c_{ij} \colon 1 \leq i \neq j \leq n\}} \mathcal{O}_F$, i.e.

$$\det(\mathrm{Stab}_H(gK)) \subset 1 + \varpi^{\min\{a_i, c_{ij} \colon 1 \leq i \neq j \leq n\}} \mathcal{O}_F.$$

The reverse inclusion is as follows: If $\min\{a_i, c_{ij} \colon 1 \leq i \neq j \leq n\} = a_k$ for some $k \leq n$, then consider for every $t \in \mathcal{O}_F$ the matrix $h_t = \mathrm{diag}(1, \ldots, 1 + \varpi^{a_k} t, \ldots, 1) \in \mathbf{GL}_n(F)$. If now $\min\{a_i, c_{ij} \colon 1 \leq i \neq j \leq n\} = c_{kl}$ for some $1 \leq k \neq l \leq n$, then consider for every $t \in \mathcal{O}_F$ the matrix $h_t$ having $1 + \varpi^{c_{kl}} t$ in the $k^{th}$ diagonal component, 1's elsewhere in the diagonal, $-\varpi^{c_{kl}} t$ in the $kl$ component and zeros everywhere else. In both cases, for every $t \in \mathcal{O}_F$, we have $h_t \in \mathrm{Stab}_H(gK)$ and $\det(h_t) = 1 + \varpi^{\min\{a_i, c_{ij} \colon 1 \leq i \neq j \leq n\}} t$, which shows the reverse inclusion. □

## VII.2.5 Local horizontal relation

We prove a local horizontal distribution relation (Corollary VII.2.5.1) using a local divisibility relation (Theorem VII.2.5.1).

### VII.2.5.1 Notation

Let $\mathbf{B} = \mathbf{B}_1 \times \mathbf{B}_2 \subset \mathbf{G}$ be the Borel subgroup with unipotent radical $\mathbf{U} = \mathbf{U}_1 \times \mathbf{U}_2$. The set of $\mathbf{B}$-antidominant diagonals is $T^- = T_1^- \times T_2^-$, where

$$\begin{aligned} T_1^- &= \left\{\operatorname{diag}(\varpi^{a_k})_{1\leq k\leq n+1}, : a_i \in \mathbb{Z} \text{ such that } a_1 \geq \cdots \geq a_{n+1}\right\} \cdot \mathbf{T}_1(\mathcal{O}_F), \\ T_2^- &= \left\{\operatorname{diag}(\varpi^{b_k})_{1\leq k\leq n}, : b_i \in \mathbb{Z} \text{ such that } b_1 \geq \cdots \geq b_n\right\} \cdot \mathbf{T}_2(\mathcal{O}_F). \end{aligned}$$

Consider the following Iwahori subgroup

$$I = I_1 \times I_2 = \{g \in K : (g \mod \varpi) \in B\}.$$

Let $\mu = \Delta \circ \mu_2 \in T$ be the cocharacter coming from the Shimura variety and denoted by $\mu_{h,v}$ in §VII.2.3.1. Here $\mu_2 \in X_*(\mathbf{T}_2)$ is given by $t \mapsto \operatorname{diag}(t, 1 \ldots, 1)$. Set $\mathbf{Frob} := \mu(\varpi)$, see §VII.2.2 for the reason why we choose this notation. Let $\mathcal{U}_\mu \in \operatorname{End}_{\mathbb{Z}[B]} \mathbb{Z}[G/K]$ be the $\mathbb{U}$-operator associated to $\mathbf{1}_{I\mu(\varpi)I} \in \mathcal{C}_c(G \mathbin{/\!/} I, \mathbb{Z})$. By Theorem IV.5.0.1, the operator $\mathcal{U}_\mu$ is annihilated by the Hecke polynomial $H_w(X) = \sum_{k=0}^{n(n+1)} A_k X^k \in (\operatorname{End}_{R[G]} R[G/K])[X]$ (see §VII.2.3.5). Set $\widetilde{H}_w(X) := H_w(q^{n-1}X)$.

Fix the classe $[1] = 1 \cdot K \in \mathbb{Z}[G/K]$ and note that for every $k \geq 1$,

$$\mathcal{U}_\mu^k([1]) = \mathcal{U}_{\mu^k}([1]) = \sum_{h \in I^+/\mu(\varpi^k)I^+\mu(\varpi^{-k})} h\,\mathbf{Frob}^k \cdot [1],$$

where, $I^+ = U \cap I$.

We fix for each element $[b]$ in $\mathcal{O}_F/\varpi\mathcal{O}_F$ a lift $b \in \mathcal{O}_F$ (e.g. Teichmuller lift). Set $S_k := \{a = \sum_i a_i \varpi^i \in \mathcal{O}_F : a_i = 0 \text{ for all } i \geq k \text{ and } [a_i] \in \mathcal{O}_F/\varpi\mathcal{O}_F \text{ for all } i < k\}$.

**Lemma** VII.2.5.1. *For every integer $k \geq 1$, the collection*

$$\left\{ (u_{k,a}, v_{k,b}) = \left( \begin{pmatrix} 1 & a_1 \cdots\cdots a_n \\ & 1 \ddots \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & b_1 \cdots\cdots b_{n-1} \\ & 1 \ddots \\ & & 1 \end{pmatrix} \right) \right\}$$

*for all $(a, b) = ((a_1, \ldots, a_n), (b_1, \ldots, b_{n-1})) \in S_k^n \times S_k^{n-1}$ forms a complete set of represen-*

*tatives for $I^+/\mu(\varpi^k)I^+\mu(\varpi^k)^{-1}$. Consequently, for every $k \geq 1$*

$$\mathcal{U}_\mu^k([1]) = \sum_{(a,b)\in S_k^n \times S_k^{n-1}} (u_{k,a}, v_{k,b})\, \boldsymbol{Frob}^k \cdot [1].$$

*Proof.* This is a consequence of the fact that $\mu(\varpi^k)I^+\mu(\varpi^{-k})$ equals

$$\begin{pmatrix} 1 & \varpi^k\mathcal{O}_F & \cdots & \varpi^k\mathcal{O}_F \\ & 1 & \mathcal{O}_F \cdots & \mathcal{O}_F \\ & & \ddots & \vdots \\ & & & \mathcal{O}_F \\ & & & 1 \end{pmatrix} \times \begin{pmatrix} 1 & \varpi^k\mathcal{O}_F & \cdots & \varpi^k\mathcal{O}_F \\ & 1 & \mathcal{O}_F \cdots & \mathcal{O}_F \\ & & \ddots & \vdots \\ & & & \mathcal{O}_F \\ & & & 1 \end{pmatrix} \subset I^+.$$

Any $g \in I_1^+$, can be written as

$$g = \begin{pmatrix} 1 & a_1 \cdots\cdots a_n \\ & \boxed{h} \end{pmatrix},$$

for some $h \in K_1$ (which is upper triangular modulo $\varpi$) and $(a_i) \in \mathcal{O}_F^n$. Therefore

$$g\mu(\varpi^k)I_1^+\mu(\varpi^{-k}) = g\iota(h^{-1})\mu(\varpi^k)I_1^+\mu(\varpi^{-k}) = u_{a',k}\mu(\varpi^k)I_1^+\mu(\varpi^{-k})$$

with some $a' = (a_i') \in \mathcal{O}_F^n$. The right action of $\mu(\varpi^k)I_1^+\mu(\varpi^{-k})$ on $I_1^+$ can only kill $\varpi^k\mathcal{O}_F$ in each factor $a_i'$. This shows that each class admits a representative of the form $u_{k,a'}$ that is unique modulo $\varpi^k$. Similar statement hold for $I_2^+$. In total, this shows that is $S_k^n \times S_k^{n-1}$ is a complete set of representatives for the quotient $\mu(\varpi^k)I^+\mu(\varpi^{-k})$. $\square$

#### VII.2.5.2 Divisibility in $\mathbb{Z}[H_0\backslash G/K]$

Consider now the following natural surjective homomorphisms of $\mathbb{Z}$-modules over the group algebra $\mathcal{H}(G /\!/ K)[H]$[16]

$$\begin{tikzcd} \mathbb{Z}[G/K] \arrow[r, two heads, "\phi"] \arrow[rr, bend left, two heads, "\phi_0"] & \mathbb{Z}[H^{\mathrm{der}}\backslash G/K] \arrow[r, two heads] & \mathbb{Z}[H_0\backslash G/K] \end{tikzcd}$$

where, $H^{\mathrm{der}} := \mathbf{H}^{\mathrm{der}}(F) = \Delta(\mathbf{SL}_n)(F)$ and $H_0 \subset H$ is the normal subgroup $\det^{-1}(\mathcal{O}_F^\times) \supset H^{\mathrm{der}}$. We derive the local horizontal distribution relations (Corollary VII.2.5.1) from the following result

[16] The $H \times \mathcal{H}(G /\!/ K)$-equivariance is seen from the identification $\mathcal{H}(G /\!/ K) \simeq \mathrm{End}_{\mathbb{Z}[G]}\, \mathbb{Z}[G /\!/ K]$.

**Theorem** VII.2.5.1 (Horizontal divisibility relation). *We have*

$$H_w(\textbf{\textit{Frob}})\cdot\phi_0([1])\equiv 0 \mod q^{n-1}(q-1)R[H_0\backslash G/K], \qquad (n\geq 1).$$

*Proof.* Recall that $\mathcal{U}_\mu\in\mathrm{End}_{\mathbb{Z}[B]}\mathbb{Z}[G/K]$, $H_w(\mathcal{U}_\mu)=0$ and $\mathbf{Frob}\in T\subset B'$. We have,

$$\begin{aligned}\widetilde{H}_w(\mathbf{Frob})\cdot\phi_0([1]) &= \phi_0\left((\widetilde{H}_w(\mathbf{Frob})-H_w(\mathcal{U}_\mu))\cdot[1]\right)\\ &= \phi_0\Big(\sum_{k=0}^{n(n+1)}A_k(q^{k(n-1)}\mathbf{Frob}^k-\mathcal{U}_\mu^k)\cdot[1]\Big)\\ &= \sum_{k=0}^{n(n+1)}A_k\phi_0\left((q^{k(n-1)}\mathbf{Frob}^k-\mathcal{U}_\mu^k)\cdot[1]\right)\end{aligned}$$

where, we have used since $A_k\in\mathrm{End}_{R[G]}R[G/K]$ for the last equality. In order to prove Theorem VII.2.5.1, it is sufficient to show the lemma below. □

**Lemma** VII.2.5.2. *For any integer $k\geq 1$ we have*

$$\phi_0\left((\mathcal{U}_\mu^k-q^{k(n-1)}\textbf{\textit{Frob}}^k)\cdot[1]\right)\equiv 0 \mod q^{k(n-1)}(q-1)\mathbb{Z}[H_0\backslash G/K].$$

*Proof.* In steps A,B and C of the proof, we will be working only mod $H^{\mathrm{der}}$, and we will wait until step D to project our calculations mod $H_0$.

A. We have,

$$\begin{aligned}\phi(\mathcal{U}_\mu^k([1])) &= \sum_{(a,b)\in S_k^n\times S_k^{n-1}}\phi((u_{k,a},v_{k,b})\,\mathbf{Frob}^k\cdot[1]) &&(\text{Lemma } VII.2.5.1)\\ &= \sum_{(a,b)\in S_k^n\times S_k^{n-1}}\phi((\iota(v_{k,b})^{-1}u_{k,a},1_n)\,\mathbf{Frob}^k\cdot[1]) &&(\Delta(v_{k,b})\in H^{\mathrm{der}})\\ &= \sum_{(a,b)\in S_k^n\times S_k^{n-1}}\phi((u_{k,a-b},1_n)\,\mathbf{Frob}^k\cdot[1]),\end{aligned}$$

where, $a-b:=(a_1-b_1,\dots,a_{n-1}-b_{n-1},a_n)\in S_k^n$ and $1_n=\mathrm{diag}(1,\dots,1)\in\mathbf{GL}_n(F)$. By substituting $c=(a-b)\in S_k^n$[17], we get

$$\phi(\mathcal{U}_\mu^k([1]))=q^{k(n-1)}\mathbf{Frob}^k\cdot\phi([1])+q^{k(n-1)}\sum_{c\in S_k^n\backslash\{0_n\}}\phi((u_{k,c},1_n)\,\mathbf{Frob}^k\cdot[1]),$$

where the first term in the right-hand-side is the contribution for $c=a-b=0$ i.e. $(a_i=b_i$ for $1\leq i\leq n-1$ and $a_n=0)$. Therefore,

$$\maltese:=\phi((\mathcal{U}_\mu^k-q^{k(n-1)}\mathbf{Frob})\cdot[1])=q^{k(n-1)}\sum_{c\in S_k^n\backslash\{0_n\}}\phi((u_{k,c},1_n)\,\mathbf{Frob}^k\cdot[1])$$

[17] We sum over $c=(a-b)$'s. For every fixed $c$ there is $q^{k(n-1)}=|(\mathcal{O}_F/\varpi^k\mathcal{O}_F)^{n-1}|$ choice of pairs $(a,b)\in(\mathcal{O}_F/\varpi^k\mathcal{O}_F)^n\times(\mathcal{O}_F/\varpi^k\mathcal{O}_F)^{n-1}$ such that $c=a-b=(a_1-b_1,\dots,a_{n-1}-b_{n-1},a_n)$.

B. Define the following map

$$\varepsilon\colon S_k^n \to (\varpi^{\mathbb{N}})^n$$

sending $c = (c_1, \ldots, c_n) \in S_k^n$ to $\underline{\varepsilon}_c := (\varepsilon(c_1), \ldots, \varepsilon(c_n))$ such that $\varepsilon(c_i) := \varpi^{\mathrm{ord}_F(c_i)}$ with $\varepsilon(0) = 0$ (note that $\mathrm{ord}_F(c_i) \leq k-1$ and that $\varepsilon(c) = 0$ if and only if $\underline{\varepsilon}_c = 0_n := (0, \ldots, 0)$).

Put $\mathcal{E} := \varepsilon(S_k^n \setminus \{0_n\})$. Note that we can view any $n$-tuple $\underline{\varepsilon} \in \mathcal{E}$ as a $n$-tuple $\underline{\varepsilon} \in S_k^n$. Set

$$\tilde{x} := \begin{cases} x/\varpi^{\mathrm{ord}_F(x)} \in \mathcal{O}_F^\times \text{ if } x \in \mathcal{O}_F \setminus \{0\}, \\ 1 \text{ if } x = 0. \end{cases}$$

For every $c = (c_i) \in S_k^n \setminus \{0_n\}$ consider the following two matrices

$$\overline{c} := \mathrm{diag}\left(\tilde{c}_n^{n-1} \prod_{i=1}^{n-1} \tilde{c}_i^{-1}, \tilde{c}_n^{-1}\tilde{c}_1, \ldots, \tilde{c}_n^{-1}\tilde{c}_i, \ldots, \tilde{c}_n^{-1}\tilde{c}_{n-1}\right) \in K_2 \cap \mathbf{SL}_n(F),$$

and

$$\underline{c} := \mathrm{diag}\left(\tilde{c}_n, \tilde{c}_n\tilde{c}_1^{-1}, \ldots, \tilde{c}_n\tilde{c}_i^{-1}, \ldots, \tilde{c}_n\tilde{c}_{n-1}^{-1}, 1\right) \in K_1.$$

Therefore, we have

$$\begin{aligned} \phi\left((u_{k,c}, 1_n)\,\mathbf{Frob}^k \cdot [1]\right) &= \phi\left(\Delta(\overline{c})(u_{k,c}, 1_n)\,\mathbf{Frob}^k \cdot [1]\right) && (\text{since } \Delta(\overline{c}) \in H^{\mathrm{der}}) \\ &= \phi\left((\iota(\overline{c})u_{k,c}\underline{c}, \overline{c})\,\mathbf{Frob}^k \cdot [1]\right) && (\text{since } \underline{c} \in K_1 \cap T_1) \\ &= \phi\left((\iota(\overline{c})u_{k,c}\underline{c}, 1_n)\,\mathbf{Frob}^k \cdot [1]\right) && (\text{since } \overline{c} \in K_2 \cap T_2) \end{aligned}$$

A direct calculation yields the equality

$$\iota(\overline{c})u_{k,c}\underline{c} = \mathrm{diag}\left(\tilde{c}_n^n \prod_{i=1}^{n-1} \tilde{c}_i^{-1}, 1, \ldots, 1\right) u_{k,\underline{\varepsilon}_c} \in \mathbf{GL}_{n+1}(F).$$

For every $c \in S_k^n \setminus \{0_n\}$, set

$$\alpha(c) := (\tilde{c}_n^n \prod_{i=1}^{n-1} \tilde{c}_i^{-1} \mod \varpi^k\mathcal{O}_F) \in (\mathcal{O}_F/\varpi^k\mathcal{O}_F)^\times.$$

The above equality shows that for every $c = (c_i), c' = (c'_i) \in S_k^n \setminus \{0_n\}$:

$$\alpha(c) = \alpha(c') \text{ and } \varepsilon(c) = \varepsilon(c') \Rightarrow \phi((u_{k,c}, 1_n)\,\mathbf{Frob}^k \cdot [1]) = \phi((u_{k,c'}, 1_n)\,\mathbf{Frob}^k \cdot [1]).$$

C. Using this we continue the computation of ✠ by regrouping terms over $c \in S_k^n \setminus \{0_n\}$ giving the same values by $\alpha$ and $\varepsilon$.

$$\begin{aligned} ✠ &= q^{k(n-1)} \sum_{c \in S_k^n \setminus \{0_n\}} \phi((u_{k,c}, 1_n)\,\mathbf{Frob}^k \cdot [1]) \\ &= q^{k(n-1)} \sum_{\underline{\varepsilon} \in \mathcal{E}} \sum_{\beta \in S_k^\times} \sum_{\{c \in S_k^n\colon \alpha(c) = \beta, \varepsilon(c) = \underline{\varepsilon}\}} \phi\left((\mathrm{diag}\,(\beta, 1, \ldots, 1)\, u_{k,\underline{\varepsilon}}, 1_n)\,\mathbf{Frob}^k \cdot [1]\right) \\ &= q^{k(n-1)} \sum_{\underline{\varepsilon} \in \mathcal{E}} \sum_{\beta \in S_k^\times} |J(\underline{\varepsilon}, \beta)|\, \phi\left((\mathrm{diag}\,(\beta, 1, \ldots, 1)u_{k,\underline{\varepsilon}}, 1_n)\,\mathbf{Frob}^k \cdot [1]\right) \end{aligned}$$

where, $J(\underline{\varepsilon}, \beta) := \{c \in S_k^n \setminus \{0_n\}\colon \alpha(c) = \beta,\, \varepsilon(c) = \underline{\varepsilon}\}$.

D. Now we project $\maltese \in \mathbb{Z}[H^{\mathrm{der}}\backslash G/K]$ into $\mathbb{Z}[H_0\backslash G/K]$. This gives us

$$\begin{aligned}\phi_0(\maltese) &= \phi_0((\mathcal{U}_\mu^k - q^{k(n-1)}\mathbf{Frob})\cdot[1]) \\ &= q^{k(n-1)}\sum_{\underline{\varepsilon}\in\mathcal{E}}\sum_{\beta\in S_k^\times}|J(\underline{\varepsilon},\beta)|\,\phi_0\left((\mathrm{diag}(\beta,1,\dots,1)u_{k,\underline{\varepsilon}},1_n)\mathbf{Frob}^k\cdot[1]\right) \\ &= q^{k(n-1)}\sum_{\underline{\varepsilon}\in\mathcal{E}}\sum_{\beta\in S_k^\times}|J(\underline{\varepsilon},\beta)|\,\phi_0\left((u_{k,\underline{\varepsilon}},\mathrm{diag}(\beta^{-1},1,\dots,1))\mathbf{Frob}^k\cdot[1]\right) \\ &= q^{k(n-1)}\sum_{\underline{\varepsilon}\in\mathcal{E}}\Big(\sum_{\beta\in S_k^\times}|J(\underline{\varepsilon},\beta)|\Big)\,\phi_0\left((u_{k,\underline{\varepsilon}},1_n)\mathbf{Frob}^k\cdot[1]\right)\end{aligned}$$

We observe that the sum $\sum_{\beta\in S_k^\times}|J(\underline{\varepsilon},\beta)|$ is equal to

$$|\{c\in S_k^n\setminus\{0_n\}\colon \varepsilon(c)=\underline{\varepsilon}|.$$

Let $c=(c_i)$ such that $\varepsilon(c)=\underline{\varepsilon}$, which means that for every $i$, we must have $\mathrm{ord}_F(c_i)=\mathrm{ord}_F(\varepsilon_i)$. It implies that for each $i$ such that $\varepsilon_i\neq 0$, the $\tilde{c}_i=c_i/\varpi^{\mathrm{ord}_F\varepsilon_i}$ are such that $\mathrm{ord}_F(\tilde{c}_i)\le k-\mathrm{ord}_F\varepsilon_i$. Hence, by definition of the set $S_k$, the set defined by these elements is described as follows

$$\left\{\sum_{j\ge 0}^{j=k-\mathrm{ord}_F\varepsilon_j} a_j\varpi^j\in\mathcal{O}_F^\times\colon a_j=0 \text{ for all } j\ge k-\mathrm{ord}_F\varepsilon_j\right\}.$$

This shows that there exist $q^{k-\mathrm{ord}_F\varepsilon_i}-q^{k-\mathrm{ord}_F\varepsilon_i-1}$ possible choices for $c_i$ for each such $i$ (with $\varepsilon_i\neq 0$). Therefore,

$$\sum_{\beta\in(\mathcal{O}_F/\varpi^k\mathcal{O}_F)^\times}|J(\underline{\varepsilon},\beta)|=\prod_i(q^{k-\mathrm{ord}_F\varepsilon_i}-q^{k-1-\mathrm{ord}_F\varepsilon_i}),$$

where the product is taken over the indices $1\le i\le n$ such that $\varepsilon_i\neq 0$. Since this set is nonempty for any $\underline{\varepsilon}\neq 0_n$, we deduce that

$$\sum_{\beta\in(\mathcal{O}_F/\varpi^k\mathcal{O}_F)^\times}|J(\underline{\varepsilon},\beta)|\equiv 0 \mod (q-1),$$

and accordingly, for all $k\ge 1$,

$$\phi_0\left((\mathcal{U}_\mu^k-q^{k(n-1)}\mathbf{Frob})\cdot[1]\right)\equiv 0 \mod q^{k(n-1)}(q-1)R[H_0\backslash G/K]. \quad \square$$

### VII.2.5.3 Local horizontal relation

Recall that $H_c=\det^{-1}(1+\varpi^c\mathcal{O}_{F_v})$, for $c\in\mathbb{N}$.

**Corollary** VII.2.5.1. *Set $x_0:=\phi([1])$. There exists $x\in R[H^{\mathrm{der}}\backslash G/K]^{H_1}$ such that*

$$H_w(\boldsymbol{Frob})\cdot x_0=\mathrm{Tr}_{1,0}\,x\in R[H^{\mathrm{der}}\backslash G/K]^{H_0},$$

*where,* $\mathrm{Tr}_{1,0}\,\phi(v):=\sum_{h\in H_0/H_1}h\cdot\phi(v)$.

*Proof.* The action of $H_0$ on $\mathbb{Z}[H^{\mathrm{der}}\backslash G/K]$ factors through $H_0/H^{\mathrm{der}}$ which we identify with $\mathcal{O}_F^\times$ through the determinant map. We note that

$$H_w(\mathbf{Frob})\cdot x_0 \in R[H^{\mathrm{der}}\backslash G/K]^{H_0},$$

since, the induced action of $\mathcal{O}_F^\times$ commutes with the induced action of $H_w(\mathbf{Frob})$ and fixes $\phi([1])$, as $\det(K\cap H)=\det K_2=\mathcal{O}_F^\times$.

Write

$$H_w(\mathbf{Frob})\cdot x_0 = \sum_{y\in H^{\mathrm{der}}\backslash G/K} a_y y,$$

with only finitely many nonzero integral coefficients $a_y\in\mathbb{Z}$. The stabilizer of any $y\in H^{\mathrm{der}}\backslash G/K$ in $H/H^{\mathrm{der}}\simeq F^\times$ is of the form $1+\varpi^{c(y)}\mathcal{O}_{F_v}$ for some integer $c(y)>0$ that we call the conductor of $y$. We have,

$$\begin{aligned}
H_w(\mathbf{Frob})\cdot x_0 &= \sum_{y\in H^{\mathrm{der}}\backslash G/K} a_y y\\
&= \sum_{c\geq 0}\sum_{\substack{y\in H^{\mathrm{der}}\backslash G/K\\ c(y)=c}} a_y y\\
&= \sum_{\substack{y\in H^{\mathrm{der}}\backslash G/K\\ c(y)=0}} a_y y + \sum_{c\geq 1}\sum_{\substack{H_0y\in H_0\backslash G/K\\ c(y)=c}} a_y \sum_{h\in H_0/H_c} h\cdot y
\end{aligned}$$

In the third equality, we may sum up over classes $H_0y$ since $H_w(\mathbf{Frob})\cdot x_0$ is $H_0$-invariant. In the last sum above, choose for each $H_0$-orbit $H_0y$ with $c_(y)\geq 1$ some $H_1$-orbit $H_1\tilde{y}\subset H_0y$. Set

$$x := \sum_{\substack{y\in H^{\mathrm{der}}\backslash G/K\\ c(y)=0}} \frac{a_y}{q-1}y + \sum_{c\geq 1}\sum_{\substack{H_0y\in H_0\backslash G/K\\ c(y)=c}} a_y \sum_{h\in H_1/H_c} h\cdot\tilde{y}.$$

By Theorem VII.2.5.1, $(q-1)\mid a_y$ if $c(y)=0$, which gives $x\in R[H^{\mathrm{der}}\backslash G/K]$ with $\mathrm{Tr}_{1,0}\,x = H_w(\mathbf{Frob})\cdot x_0$. □

**Remark** VII.2.5.1. *We will denote the element $x$ constructed above by $x_v$ to keep track of its associated place $v\in\mathcal{P}_{sp}$.*

# VII.3 Split distribution relations

## VII.3.1 Definition of the norm-compatible system

For any $\mathfrak{f} \in \mathcal{N}_{sp}^{r}$, set $\mathcal{P}_{\mathfrak{f}} \subset \mathcal{P}_{sc}$ for the places of $F$ defined by the prime ideals dividing $\mathfrak{f}$. Define

$$\xi_{\mathfrak{f}} := u(r)^{-1} \cdot \pi_{\mathrm{cyc}}\big([g_{0,S}] \otimes [1]^{S}(\mathfrak{f})\big) \in \mathbb{Q}[\mathcal{Z}_{\mathbf{G},K}(\mathbf{H})]$$

where, $[g_{0,S}] := \mathbf{H}^{\mathrm{der}}(F_S) g_{0,S} K_S$, $[1]^{S}(\mathfrak{f}) := \big(\otimes_{v \in \mathcal{P}_{\mathfrak{f}}} x_v\big) \otimes \big(\otimes_{v \notin S \cup \mathcal{P}_{\mathfrak{f}}} [1]_v\big)$ and

$$u(r) = \begin{cases} [E^{\times} \cap \mathbb{A}_{F,f}^{\times} \mathfrak{O}_1^{\times} : F^{\times}] & \text{If } r = 0, \\ 1 & \text{If } r \geq 1. \end{cases}$$

**Proposition** VII.3.1.1. *For each $\mathfrak{f} \in \mathcal{N}_{sp}$, the field of definition $E_{\mathfrak{f}}$ of $\xi_{\mathfrak{f}}$ is contained in $\mathcal{K}(\mathfrak{f})$ the $\mathcal{K}$-transfer field of conductor $\mathfrak{f}$.*

*Proof.* Write $\xi_{\mathfrak{f}} = \sum a_i \mathfrak{z}_{g_i}$ $(a_i \in \mathbb{Q})$ with

$$g_{i,S} = g_{0,S}, \ \forall v \notin S \cup \mathcal{P}_{\mathfrak{f}} \quad g_{i,v} = g_{0,v}$$

and $x_v = \sum_i a_i [g_i]_v$ for all $v \in \mathcal{P}_{\mathfrak{f}}$. The stabilizer of $\mathfrak{z}_{g_i}$ in $\mathbf{H}(\mathbb{A}_f)$ contains (as in Lemma VII.1.3.1)

$$\mathbf{H}(\mathbb{Q}) \cdot \left( K_{\mathbf{H},g_0,S}^{Z} \times K_{\mathbf{H}}^{S \cup \mathcal{P}_{\mathfrak{f}}} \times \prod_{\mathfrak{p} \in \mathcal{P}_{\mathfrak{f}}} K_{i,\mathfrak{p}} \right),$$

for some open compact subgroups $K_{i,\mathfrak{p}} \subset K_{\mathfrak{p}}$. Therefore, the stabilizer of $\xi_{\mathfrak{f}}$ in $\mathbf{T}^1(\mathbb{A}_f)$ contains[18]

$$\mathbf{T}^1(\mathbb{Q}) \cdot \left( U_{g_0,S} \times U_{\mathfrak{f}}^{S \cup \mathcal{P}_{\mathfrak{f}}} \times \prod_{v \in \mathcal{P}_{\mathfrak{f}}} U_v^1(1) \right) = \mathbf{T}^1(\mathbb{Q}) \cdot U_{\mathfrak{f}},$$

and accordingly, $E_{g_{\mathfrak{f}}}$ is contained in $\mathcal{K}(\mathfrak{f})$ for which we have (see §VII.1.5)

$$\mathrm{Gal}(\mathcal{K}(\mathfrak{f})/E) \simeq \frac{\mathbf{T}^1(\mathbb{A}_f)}{\mathbf{T}^1(\mathbb{Q}) U_{\mathfrak{f}}} \simeq \frac{\mathbb{A}_{E,f}^{\times}}{E^{\times} \mathbb{A}_{F,f}^{\times} \mathfrak{O}_{\mathfrak{f}}^{\times}}. \quad \square$$

**Remark** VII.3.1.1. *Proposition VII.3.1.1 and Lemma VII.1.5.2 imply that $E_{\mathfrak{f}}$ is also contained in $E(\mathfrak{c}_1 \mathfrak{f})$. Therefore, by Corollary VII.1.5.1 any prime ideal of $E$ above an ideal $\mathfrak{p} \in \mathcal{P}$ that is prime to $\mathfrak{f}$ is unramified in the extension $E_{\mathfrak{f}}/E$, i.e.*

$$E_{\mathfrak{f}} \subset \mathcal{K}(\mathfrak{f}) \subset E(\infty)^{un, w_{\mathfrak{p}}}.$$

[18] Since by Corollary VII.2.5.1, we have $[x_v] \in (\mathbb{Z}[q_v^{\pm 1}][H_v^{\mathrm{der}} \backslash G_v / K_v])^{U_v^1(1)}$.

### VII.3.2 Proof of Theorem VII.1.9.2

For any $\mathfrak{f} \in \mathcal{N}_{sp}^{r}$ and any place $v_\circ \in \mathcal{P}_{sc} \setminus \mathcal{P}_{\mathfrak{f}}$ with prime ideal $\mathfrak{p}_{v_\circ} \in \mathcal{P}_{sp}$, we show here that:

$$H_{w_\circ}(\mathrm{Fr}_{w_\circ}) \cdot \xi_{\mathfrak{f}} = \mathrm{Tr}_{\mathcal{K}(\mathfrak{p}_{v_\circ}\mathfrak{f})/\mathcal{K}(\mathfrak{f})}\, \xi_{\mathfrak{p}_{v_\circ}\mathfrak{f}},$$

where, $H_{w_\circ} \in \mathcal{H}_{K_{\mathfrak{p}_{v_\circ}}}(\mathbb{Z}[q_{v_\circ}^{\pm 1}])[X]$ is the Hecke polynomial attached to $\mathrm{Sh}_K(\mathbf{G}, \mathcal{X})$ at the place $w_\circ$ of the reflex field $E = E(\mathbf{G}, \mathcal{X})$ defined by $\iota_{v_\circ}$. Indeed, $H_{w_\circ}(\mathrm{Fr}_{w_\circ})\xi_{\mathfrak{f}}$ is equal to:

$$\begin{aligned}
&= u(r)^{-1}\pi_{\mathrm{cyc}}\big([g_{0,s}] \otimes (H_{w_\circ}(\mathrm{Fr}_{w_\circ})[1]_{v_\circ}) \otimes (\otimes_{v\in\mathcal{P}_{\mathfrak{f}}} x_v) \otimes \big(\otimes_{v\notin S\cup\mathcal{P}_{\mathfrak{f}}\cup\{v_\circ\}}[1]_v\big)\,\big)\\
&\overset{(\text{Cor. VII.2.5.1})}{=} u(r)^{-1}\pi_{\mathrm{cyc}}\big([g_{0,s}] \otimes \big(\sum_{\lambda\in\mathcal{O}_{v_\circ,0}^{\times}/\mathcal{O}_{v_\circ,1}^{\times}} \lambda\cdot x_{v_\circ}\big) \otimes (\otimes_{v\in\mathcal{P}_{\mathfrak{f}}} x_v) \otimes \big(\otimes_{v\notin S\cup\mathcal{P}_{\mathfrak{f}}\cup\{v_\circ\}}[1]_v\big)\,\big)\\
&= u(r)^{-1}\sum_{\lambda\in\mathcal{O}_{v_\circ,0}^{\times}/\mathcal{O}_{v_\circ,1}^{\times}} \pi_{\mathrm{cyc}}\big([g_{0,s}] \otimes (\lambda\cdot x_{v_\circ}) \otimes (\otimes_{v\in\mathcal{P}_{\mathfrak{f}}} x_v) \otimes \big(\otimes_{v\notin S\cup\mathcal{P}_{\mathfrak{f}}\cup\{v_\circ\}}[1]_v\big)\,\big)\\
&= u(r)^{-1}\sum_{\lambda\in\mathcal{O}_{v_\circ,0}^{\times}/\mathcal{O}_{v_\circ,1}^{\times}} \cdot\pi_{\mathrm{cyc}}\big(\lambda\cdot([g_{0,s}] \otimes (\otimes_{v\in\mathcal{P}_{\mathfrak{f}}\cup v_\circ} x_v) \otimes \big(\otimes_{v\notin S\cup\mathcal{P}_{\mathfrak{f}}\cup\{v_\circ\}}[1]_v\big))\big)\\
&\overset{(\text{Pro. VI.15.0.1})}{=} u(r)^{-1}\sum_{\lambda\in\mathcal{O}_{v_\circ,0}^{\times}/\mathcal{O}_{v_\circ,1}^{\times}} \cdot\pi_{\mathrm{cyc}}\big([g_{0,s}] \otimes (\otimes_{v\in\mathcal{P}_{\mathfrak{f}}\cup v_\circ} x_v) \otimes \big(\otimes_{v\notin S\cup\mathcal{P}_{\mathfrak{f}}\cup\{v_\circ\}}[1]_v\big)\,\big)^{\mathrm{Art}_{w_\circ}(\lambda)}\\
&= u(r)^{-1}\sum_{\lambda\in\mathcal{O}_{v_\circ,0}^{\times}/\mathcal{O}_{v_\circ,1}^{\times}} \xi_{\mathfrak{p}_{v_\circ}\mathfrak{f}}^{\mathrm{Art}_{w_\circ}(\lambda)}\\
&= u(r)^{-1}\,\mathrm{Tr}_{\mathcal{K}(\mathfrak{p}_{v_\circ}\mathfrak{f})_{w_\circ}/\mathcal{K}(\mathfrak{f})_{w_\circ}}\, \xi_{\mathfrak{p}_{v_\circ}\mathfrak{f}}\\
&\overset{(\text{Prop. VII.1.7.1})}{=} \sum_{\sigma\in\mathrm{Gal}(\mathcal{K}(\mathfrak{p}_{v_\circ}\mathfrak{f})/\mathcal{K}(\mathfrak{f}))} \xi_{\mathfrak{p}_{v_\circ}\mathfrak{f}}^{\sigma}\\
&= \mathrm{Tr}_{\mathcal{K}(\mathfrak{p}_{v_\circ}\mathfrak{f})/\mathcal{K}(\mathfrak{f})}\, \xi_{\mathfrak{p}_{v_\circ}\mathfrak{f}},
\end{aligned}$$

where, $\mathrm{Art}_{w_\circ}\colon E_{w_\circ} \hookrightarrow \mathbb{A}_{E,f}^{\times} \xrightarrow{\mathrm{Art}_E} \mathrm{Gal}(E^{ab}/E)$. □

**Remark** VII.3.2.1. *The methods used in the sections VII.2.5 to prove Theorem VII.1.9.2 for split places, although seemingly too technical, hide a pattern that gives, mutatis mutandis, local norm-compatible systems for inert places of $F$ too. The details of these calculations for the split vertical case and tame/vertical inert cases will appear in a forthcoming paper, which will also include similar treatment for other embeddings of Shimura data.*

# NOMENCLATURE

## Chapter II §1 & §2



## Chapter II §3

**Chapter III**

**Chapter VII**

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