Seed Relations
for
Eichler–Shimura congruences and Euler systems

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Abstract

This paper proves that the $\mathbb{U}$-operator [Bou21d] attached to a cocharacter is a right root of the corresponding Hecke polynomial. This result is an important ingredient in the proof of (i) the horizontal norm relations in the context of Gross–Gan–Prasad cycles and of (ii) the generalization of Eichler–Shimura relations.

1. Introduction

1.1 Origin of the problem

Let F be any p-adic field, q the size of its residue field and ϖ a fixed uniformizer in the ring of integers $\mathcal{O}_F$. Let $\mathcal{T}$ be the Bruhat–Tits building of GL_2 over F. This is a connected tree in which every node has $q+1$ neighbours. Let $\circ \in \mathcal{T}$ be a hyperspecial point fixed, K the maximal parahoric subgroup attached to $\circ$ and $\mathcal{T}^\circ := \{v \in GL_2(F) \cdot \circ \setminus \circ\}$.

The Hecke algebra $\mathcal{H}(GL_2(F) /\!\!/ K, \mathbb{Z})$ acts on $\mathcal{T}^\circ$ via adjacency operators, in particular the basic element $T_p := \mathbf{1}_{K \mathrm{diag}(\varpi,1) K}$, sends a vertex to the formal sum of its neighbours.

Define the operator $u_\circ \in \mathrm{End}_{\mathbb{Z}[K]} \mathbb{Z}[\mathcal{T}^\circ]$ which sends a vertex $v \neq \circ$ to its successors with respect to the origin $\circ$, in other words

$$u_\circ(v) = \sum_{\mathrm{dist}(v',\circ)=\mathrm{dist}(v,\circ)+1} v'.$$

Define also the predecessor operator $v_\circ \in \mathrm{End}_{\mathbb{Z}[K]} \mathbb{Z}[\mathcal{T}^\circ]$ sending a vertex $v \neq \circ$ to the unique $v' \in [\circ, v]$ verifying $\mathrm{dist}(v', \circ) = \mathrm{dist}(v, \circ) - 1$. The operators $u_\circ$ verify the following properties

$$v_\circ \circ u_\circ = p\, \mathrm{Id}_{\mathcal{T}^\circ} \neq u_\circ \circ v_\circ \text{ and } T_p = u_\circ + v_\circ.$$

We insist on the fact that the operators T_p, $v_\circ$ and $u_\circ$ do not commute. An immediate consequence of these basic properties is that $u_\mathfrak{p}$ (resp. $v_\mathfrak{p}$) is a right (resp. left) root of the Hecke polynomial $H_p(X) = X^2 - T_p X + p$, i.e.

$$u_\circ^2 - T_p \circ u_\circ + p = 0 \text{ and } v_\circ^2 - v_\circ \circ T_p + p = 0.$$

The goal of this paper is to genralize such integrality relation to general unramified groups.

2020 Mathematics Subject Classification 11E95, 11G18, 20E42, 20G25 and 20C08 (primary).

The author was supported by the Swiss National Science Foundation grant #PP00P2-144658 and #P2ELP2-191672.

1.2 Motivation

In 1954, Eichler discovered the first instance of the link between zeta functions of Shimura varieties and automorphic L-functions. Shortly thereafter, Shimura extended Eichler results to compute the zeta functions of quaternionic curves. Their work was based on the congruence relation, known now as the Eichler–Shimura relation, which played an important role in the theory of arithmetic of elliptic curves and modular forms. Later on, in the 70s, Langlands launched a program that aims to generalize the previous work to compute zeta functions attached to all Shimura varieties. Gradually a conjecture generalizing the Eichler–Shimura relation has emerged and was formulated by Blasisus and Rogawski [BR94, §6]. We give its statement below after setting some background.

Let $\mathbf{G}$ be a connected, reductive group defined over $\mathbb{Q}$ and let $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}}$. Suppose we have a homomorphism of algebraic $\mathbb{R}$-groups $\mathbb{S} \to \mathbf{G}_{\mathbb{R}}$, which satisfies the axioms of Deligne [Mil17, Definition 5.5]. Let K be an open compact subgroup of $\mathbf{G}(\mathbb{A}_f)$ of the form $\prod_{v<\infty} K_v$, where $K_v \subset \mathbf{G}(\mathbb{Q}_v)$ and K_v is hyperspecial for almost all the finite places v. This gives rise to the Shimura variety $Sh_K(\mathbf{G}, \mathcal{X})$ with reflex field E and whose complex points are

$$Sh_K(\mathbf{G}, \mathcal{X})(\mathbb{C}) = \mathbf{G}(\mathbb{Q})\backslash \mathcal{X} \times \mathbf{G}(\mathbb{A}_f)/K.$$

Assume that K is sufficiently small, so that $Sh_K(\mathbf{G}, \mathcal{X})$ is a smooth. We fix a prime p over which $\mathbf{G}$ is unramified and the level structure K has the form K^pK_p with K_p hyperspecial. For each prime ideal $\mathfrak{p}$ of E lying over p, Blasius and Rogawski have defined a polynomial $H_{\mathfrak{p}} \in \mathcal{H}(\mathbf{G}(\mathbb{Q}_p)//K_p, \mathbb{Q})[X]$, and they conjectured that:

Conjecture 1.1 Blasius–Rogawski. Let ℓ be a prime $\neq p$ (i) The Shimura variety $Sh_K(\mathbf{G}, \mathcal{X})$ has good reduction at $\mathfrak{p}$ (in some sense); and (ii) we have

$$H_{\mathfrak{p}}(Fr_{\mathfrak{p}}) = 0 \text{ in the ring } End_{\mathbb{Q}_\ell}(H^{\bullet}_{\text{ét}}(Sh_K(\mathbf{G}, \mathcal{X}) \times_E \overline{\mathbb{Q}}, \mathbb{Q}_\ell)).$$

This conjecture was proved by Ihara (extending cases treated by Eichler and Shimura) for Shimura curves. The first statement has been established for Shimura varieties of abelian type by Kisin [Kis09, Kis10] and the second part was proved by: Bültel for certain orthogonal groups [Bül97], Wedhorn [Wed00] in the PEL case for groups that are split over $\mathbb{Q}_p$, Bültel–Wedhorn for the unitary case of signature $(n-1,1)$ with n even [BW06], Koskivirta for a unitary similitude group of signature $(n-1,1)$ over $\mathbb{Q}$ when p is inert in the reflex field and n odd [Kos13] and finally H. Li showed recently the conjecture for simple GSpin Shimura varieties [Li18].

In all these cases for which the conjecture is known, the authors prove a slightly stronger version of it where the desired annihilation is taking place in a "geometric" ring of correspondences in characteristic p. Assume that $Sh_K(\mathbf{G}, \mathcal{X})$ is of Hodge-type and let $\mathcal{S}_K$ be its integral model over $\mathcal{O}_{E_{\mathfrak{p}}}$. This scheme has an interpretation as a moduli space of abelian schemes with additional structures. Following Chai–Faltings [FC90], Moonen defines in [Moo04] a stack $p-$Isog over $\mathcal{O}_{E_{\mathfrak{p}}}$, parametrizing p-isogenies between two points of $\mathcal{S}_K$. It has two natural projections to $\mathcal{S}_K$, sending an isogeny to its target and source. the subalgebra generated by the irreducible components. Consider the $\mathbb{Q}$-algebra of cycles $\mathbb{Q}[p - \text{Isog} \times E]$ and $\mathbb{Q}[p - \text{Isog} \times k_{\mathcal{O}_{E_{\mathfrak{p}}}}]$ where $k_{\mathcal{O}_{E_{\mathfrak{p}}}}$ is the residue field of $\mathcal{O}_{E_{\mathfrak{p}}}$, here multiplication is defined by composition of isogenies. Define $p - \text{Isog}^{\text{ord}} \times k_{\mathcal{O}_{E_{\mathfrak{p}}}}$ as the preimage of the μ-ordinary locus of the special fiber of the $\mathcal{S}_K$, under the source projection. We get a diagram of $\mathbb{Q}$-algebra homomorphism

$$\begin{tikzcd}
\mathcal{H}(\mathbf{G}(\mathbb{Q}_p)//K_p, \mathbb{Q}) \arrow[r, "h"] \arrow[dd, "\dot{\mathcal{S}}^G_M"] & \mathbb{Q}[p - \mathrm{Isog} \times E] \arrow[d, "\sigma"] \\
 & \mathbb{Q}[p - \mathrm{Isog} \times k_{\mathcal{O}_{E_\mathfrak{p}}}] \arrow[d, shift left, "\mathrm{ord}"] \\
\mathcal{H}(\mathbf{M}(\mathbb{Q}_p)//K_p \cap M(\mathbb{Q}_p), \mathbb{Q}) \arrow[r, "\bar{h}"] & \mathbb{Q}[p - \mathrm{Isog}^{\mathrm{ord}} \times k_{\mathcal{O}_{E_\mathfrak{p}}}] \arrow[u, shift left, "\mathrm{cl}"]
\end{tikzcd}$$

where the big square is commutative, $\mathbf{M}$ is the centralizer of the norm of the dominant coweight μ given by the Shimura datum, the homomorphism $\dot{\mathcal{S}}^G_M$ is the twisted Satake transform, σ is the specialization map of cycles, the map ord intersects a cycle with the ordinary μ-locus while cl is the map sending a cycle to its closure. There is a natural Frobenius section of the source projection, mapping an abelian variety to its Frobenius isogeny, which produces a closed subscheme F of $p - \mathrm{Isog} \times k_{\mathcal{O}_{E_\mathfrak{p}}}$.

Conjecture 1.2. The cycle F is a root of the polynomial

$$\sigma \circ h(H_\mathfrak{p})(X) \in \mathbb{Q}[p - \mathrm{Isog} \times k_{\mathcal{O}_{E_\mathfrak{p}}}][X].$$

Functorial properties of cohomology shows that Conjecture 1.2 implies Conjecture 1.1. Most known cases of Conjecture 1.2 are obtained by proving first the conjecture on the generically ordinary p-isogenies. This reduces to Bültel's group theoretic result which says that we have an annihilation

$$H_\mathfrak{p}(\mu) = 0 \text{ in the } \mathbb{Q}\text{-algebra } \mathcal{H}(\mathbf{M}(\mathbb{Q}_p)//K_p \cap \mathbf{M}(\mathbb{Q}_p), \mathbb{Q}). \tag{$\star$}$$

Now, If the ordinary locus $p - \mathrm{Isog}^{\mathrm{ord}} \times k_{\mathcal{O}_{E_\mathfrak{p}}}$ is dense in $p - \mathrm{Isog} \times k_{\mathcal{O}_{E_\mathfrak{p}}}$, then Bültel's argument is sufficient to prove the full congruence conjecture. This is the cases studied by Chai-Faltings, Bültel, Wedhorn and Bültel–Wedhorn.

We have a commutative diagram:

$$\begin{tikzcd}
\mathcal{H}_K(\mathbb{Q}) \arrow[rr, hook, "\dot{\mathcal{S}}^G_M"] \arrow[d, equal] & & \mathcal{H}(\mathbf{M}(\mathbb{Q}_p) /\!/ K_p \cap \mathbf{M}(\mathbb{Q}_p), \mathbb{Q}) \arrow[d, equal] \\
End_{\mathbb{Z}[G]}\mathbb{Q}[G/K] \arrow[r, hook] & End_{\mathbb{Q}[P]}\mathbb{Q}[G/K] \arrow[r] & End_{\mathbb{Q}[\mathbf{M}(\mathbb{Q}_p)]}\mathbb{Q}[\mathbf{M}(\mathbb{Q}_p)/K_p \cap \mathbf{M}(\mathbb{Q}_p)].
\end{tikzcd}$$

Our main results (Theorem 6.4) shows in particular that Bultel's relation ($\star$) lifts naturally to an analogous relation

$$H_\mathfrak{p}(u_\mu) = 0 \in End_{\mathbb{Q}[\mathbf{P}(\mathbb{Q}_p)]}\mathbb{Q}[\mathbf{G}(\mathbb{Q}_p)/K_p] \tag{$\dagger$}$$

where u_μ is the $\mathbb{U}$-operator attached to ϖ^μ [Bou21d] and $\mathbf{P}$ is the largest parabolic subgroup of $\mathbf{G}$ relative to which μ is dominant. For applications, a key advantage of the latter relations (upon Bültel's) is that while $\mathcal{H}(\mathbf{M}(\mathbb{Q}_p)//K_p \cap \mathbf{M}(\mathbb{Q}_p), \mathbb{C})$ still had to be made acting on various spaces, the non-commutative ring $End_{\mathbb{Z}[\mathbf{P}(\mathbb{Q}_p)]}(\mathbf{G}(\mathbb{Q}_p)/K_p)$ already acts (faithfully and by definition) on the ubiquitous space $\mathbb{Q}[\mathbf{G}(\mathbb{Q}_p)/K_p]$.

In a work in progress the author is tackling (using † instead) a generalization of Conjecture 1.2 for abelian-type Shimura varieties [Bou21c].

1.3 Main result

Let F be a finite extension of $\mathbb{Q}_p$ for some prime p, $\mathcal{O}_F$ its ring of integers, ϖ a fixed uniformizer in $\mathcal{O}_F$ and k_F the residue field of F of size q. For every scheme X over $\operatorname{Spec}\mathcal{O}_F$, we set $X_{\kappa(F)} := X \times_{\operatorname{Spec}\mathcal{O}_F} \operatorname{Spec}\kappa(F)$ for the special fiber.

Let $\mathbf{G}/F$ be an unramified reductive group, $\mathbf{S}$ a maximal F-split subtorus of $\mathbf{G}$ and $\mathcal{A}$ the apartment attached to $\mathbf{S}$ in the extended Bruhat–Tits building of $\mathbf{G}$, together with a fixed origin a hyperspecial point $a_\circ \in \mathcal{A}$. Let $\mathbf{T}$ be the centralizer of $\mathbf{S}$, which is a maximal F-torus in $\mathbf{G}$, $\mathbf{N} = N_{\mathbf{G}}(\mathbf{S})$, $\mathbf{B} = \mathbf{T}\cdot\mathbf{U}^+$ a Borel subgroup with unipotent radical $\mathbf{U}^+$ and $W = \mathbf{N}(F)/\mathbf{T}(F)$ be the Weyl group.

Let K be a hyperspecial maximal open compact subgroup of $\mathbf{G}$ attached $a_\circ$. Bruhat and Tits attach to $a_\circ$ a reductive $\mathcal{O}_F$-model $\mathcal{G}$ of $\mathbf{G}$. Let K be the corresponding parahoric subgroup, i.e. $\mathcal{G}(\mathcal{O}_F)$. This also applies to the reductive group $\mathbf{T}$ and $a_\circ$, we get then a reductive $\mathcal{O}_F$-model $\mathcal{T}$ of $\mathbf{T}$. Let I be the Iwahori subgroup that is defined by

$$I = \{g \in \mathbf{G}(\mathcal{O}_F) : red(g) \in \mathbf{B}(k_F)\}.$$

For any algebraic F-groups $\mathbf{H}$ (bold style), we denote its group of F-points by the ordinary capital letter $H = \mathbf{H}(F)$.

Let $\nu'_N \colon N \to (X_*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{R}) \rtimes W$ be the map characterized by

$$\nu'_N(\varpi^\lambda) = \lambda.$$

Note that $\nu'_N = -\nu_N$, where ν_N is the Bruhat–Tits translation homomorphism. Set

$$T_1 := \mathcal{T}(\mathcal{O}_F) = \ker\nu_N = \ker\kappa_{\mathbf{T}},$$

where $\kappa_{\mathbf{T}}$ is the Kottwitz homomorphism[1]. We embed $X_*(\mathbf{S})$ into T (using ν'_N) by identifying $\lambda \in X_*(\mathbf{S})$ with $\varpi^\lambda := \lambda(\varpi)$. Using this identification, we have

$$\Lambda_T := T/T_1 \simeq X_*(\mathbf{T})_F \simeq X_*(\mathbf{S}).$$

Set Φ^+ for the set of B-positive roots, the one that appears in $\operatorname{Lie}(B)$, or equivalently if it takes positive values on the vectorial chamber $\mathcal{C}^-$ opposite to $\mathcal{C}^+$; where $\mathcal{C}^+$ is the vectorial chamber corresponding to B[2].

We say that $\lambda \in X_*(\mathbf{S})$ is $\mathbf{B}$-dominant if $\langle\lambda,\alpha\rangle \geqslant 0$ for all $\alpha \in \Phi^+$. Let $\overline{\mathcal{C}} \subset \mathcal{A}_{ext}$ denotes the closed vectorial chamber corresponding to the Borel B in the extended apartment attached to $\mathbf{S}$. Thus, an element $t = \varpi^\lambda$ for $\lambda \in X_*(\mathbf{S})$ is antidominant if and only if $\lambda \in X_*(\mathbf{S}) \cap \overline{\mathcal{C}}$, if and only if λ is $\mathbf{B}$-dominant, since $\langle\nu'_N(t),\alpha\rangle = \langle\lambda,\alpha\rangle \leqslant 0, \forall\alpha \in \Phi^+$. Write Λ_T^- for the set of antidominant elements in Λ_T.

For any extension E of F, let $\mathcal{M}(E)$ be the set of $\mathbf{G}(E)$-conjugacy classes of (algebraic group) cocharacters $\mathbb{G}_{m,E} \to \mathbf{G}_E$. By [Kot84a, Lemma 1.1.3], the canonical surjective morphism $X_*(\mathbf{S}) \to \mathcal{M}(F)$ yields the following identification

$$X_*(\mathbf{S})/W(\mathbf{G},\mathbf{S}) \simeq \mathcal{M}(F) \simeq \mathcal{M}(\overline{F})^{\operatorname{Gal}(\overline{F}/F)} \simeq \left(X_*(\mathbf{T})/W(\mathbf{G}_{\overline{F}},\mathbf{T})\right)^{\operatorname{Gal}(\overline{F}/F)}.$$

In addition, using the Cartan decomposition one gets another identification identification

$$\mathcal{M}(F) \simeq K\backslash G/K,$$

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[1] Note that in this unramified case, $\mathbf{T}$ splits over the completion of F^{un} denoted previously by L. Thus, the Kottwitz homomorphism takes the simpler form $\kappa_{\mathbf{T}} \colon \mathbf{T}(L) \to X_*(\mathbf{T})$.

[2] Given B, the chamber $\mathcal{C}^+$ is the unique vectorial chamber with apex $a_\circ$ for which T_1U^+ is the union of the fixators of all quartiers $a + \mathcal{C}^+$ with $a \in \mathcal{A}$.

given by $[\lambda] \mapsto K\varpi^\lambda K$.

Let $\mathfrak{c} \in \mathcal{M}(\overline{F})$ and $F(\mathfrak{c}) \subset F^{un}$ its field of definition. Set $d = [F(\mathfrak{c}) : F]$. Let $\mu \in Norm_{F(\mathfrak{c})/F}\mathfrak{c}$ be the cocharacter of $\mathbf{T}$ which is $\mathbf{B}$-dominant, i.e. ϖ^μ is antidominant. Let $\mathbf{P}$ be the largest parabolic subgroup of $\mathbf{G}$ relative to which μ is dominant, $\mathbf{L}$ is a Levi factor of $\mathbf{P}$ (which is also the centralizer of μ in $\mathbf{G}$) and $\mathbf{U}_P^+$ the unipotent radical of $\mathbf{P}$.

In [Bou21d], to any element $t \in \Lambda_T^-$ is attached an operator $u_t \in End_{\mathbb{Z}[B]}\mathbb{Z}[G/K]$ characterized by sending the trivial class K to $\sum_{u \in I/I\cap tIt^{-1}} utK$ (and extended B-equivariantly to $\mathbb{Z}[G/K]$).

The main result of the paper (which generalizes [BBJ18, Lemma 3.3]) is:

THEOREM 1.3 Seed relation. *The operator $u_{\varpi^\mu} \in \mathbb{U}$ is a right root of the Hecke polynomial $H_{\mathbf{G},\mathfrak{c}}$ in $End_{\mathbb{Z}[P]}\mathbb{Z}[q^{\pm 1}][G/K]$.*

REMARK 1.4. The minimal polynomial of u_{ϖ^μ} has actually its coefficients in the integral Hecke algebra $End_{\mathbb{Z}[G]}\mathbb{Z}[G/K]$.

REMARK 1.5. This relation has another application; in [Bou21b] (resp. [BBJ18]) we construct a tame (resp. vertical) norm compatible system of special cycles in a (product of) unitary Shimura variety.

REMARK 1.6. A very interesting and surprising aspect of this work is that in order to establish formulas relating the two non-commuting commutative subrings, $\mathbb{U}$ and $\mathcal{H}_K(G)$, of the Hecke algebra $End_{\mathbb{Z}[B]}(\mathbb{Z}[q^{\pm 1}][G \mathbin{/\!\!/} K])$ one has to embed them both in yet another noncommutative ring (the Iwahori–Hecke algebra $\mathcal{H}_I(\mathbb{Z}[q^{-1}])$), where they actually do commute!

1.4 Acknowledgement

Some parts of this article originated from my doctoral thesis, directed by D. Jetchev, to whom I am very grateful. I am thankful to C. Cornut for his support and meticulous reading and to T. Wedhorn; the discerning reader will no doubt notice the importance of his paper [Wed00].

2. Langlands dual group

Let $\Gamma_{un} = \mathrm{Gal}(F^{un}/F) \simeq \mathrm{Gal}(\overline{k}_F/k_F)$. As before, we let $\sigma \in \Gamma_{un}$ be the arithmetic Frobenius of F. The group $\mathbf{G}$ split over F^{un} [GD70, XXVI 7.15]. We consider a Langlands dual group of $\mathbf{G}$ with respect to Γ_{un}. This group sits in the following short exact sequence

$$\begin{tikzcd} 1 \arrow[r] & \widehat{\mathbf{G}} \arrow[r] & {}^L\mathbf{G} \arrow[r] & \Gamma_{un} \arrow[r] & 1, \end{tikzcd}$$

and every choice of épinglage $(\widehat{\mathbf{B}}, \widehat{\mathbf{T}}, (e_\alpha))$[3] yields a splitting of the above exact sequence. We fix a Γ_{un}-invariant épinglage [Kot84b, §1] thus ${}^L\mathbf{G} = \widehat{\mathbf{G}} \rtimes \Gamma_{un}$.

The Γ_{un}-equivariant isomorphism $X_*(\mathbf{T}) \simeq X^*(\widehat{\mathbf{T}})$ induces a canonical identification between the Γ_{un}-groups $W(\mathbf{G}_{\overline{F}}, \mathbf{T})$ and the Weyl group $W(\widehat{\mathbf{G}}, \widehat{\mathbf{T}})$ and an identification between the $X_*(\mathbf{S}) = X_*(\mathbf{T})_F$ and $X^*(\widehat{\mathbf{S}})$. The inclusion $\mathbf{S} \hookrightarrow \mathbf{T}$ gives an embedding $X_*(\mathbf{S}) \hookrightarrow X_*(\mathbf{T})$,which yields a short exact sequence

$$\begin{tikzcd} 1 \arrow[r] & \widehat{\mathbf{T}}^{1-\sigma} \arrow[r] & \widehat{\mathbf{T}} \arrow[r] & \widehat{\mathbf{S}} \arrow[r] & 1\ , \end{tikzcd}$$

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[3] Here, for each simple root α of $\widehat{\mathbf{T}}$, e_α is a nonzero element of the root vector space $\mathrm{Lie}(\widehat{\mathbf{G}})_\alpha$.

showing that $\widehat{\mathbf{S}} \simeq \widehat{\mathbf{T}}/(1-\sigma)\widehat{\mathbf{T}}$. Therefore,

$$\widehat{\mathbf{T}} = \mathrm{Spec}(\mathbb{C}[X^*(\widehat{\mathbf{T}})]) = \mathrm{Spec}(\mathbb{C}[X_*(\mathbf{T})]),$$

$$\widehat{\mathbf{S}} = \mathrm{Spec}(\mathbb{C}[X_*(\mathbf{S})]) = \mathrm{Spec}(\mathbb{C}[\Lambda_T]) = \mathrm{Spec}\left(\mathcal{C}_c(\mathbf{T}(F) /\!\!/ \mathcal{T}(\mathcal{O}_F), \mathbb{C})\right).$$

In particular, $\widehat{\mathbf{S}}(\mathbb{C}) = \mathrm{Hom}(X_*(\mathbf{T})_F, \mathbb{C}^\times)$. The above identification $W(\mathbf{G}_{\overline{F}}, \mathbf{T}) \simeq W(\widehat{\mathbf{G}}, \widehat{\mathbf{T}})$, lets $W(\mathbf{G}, \mathbf{S})$ operates on $\widehat{\mathbf{S}}$ by duality. The space $\widehat{\mathbf{S}}/W(\mathbf{G}, \mathbf{S})$ has the structure of a smooth affine $\mathbb{C}$-scheme whose coordinate ring is $\mathbb{C}[X_*(\mathbf{S})]^{W(\mathbf{G},\mathbf{S})}$:

$$\widehat{\mathbf{S}}/W(\mathbf{G}, \mathbf{S}) = \mathrm{Spec}\left(\mathbb{C}[X_*(\mathbf{S})]^{W(\mathbf{G},\mathbf{S})}\right) = \mathrm{Spec}\left(\mathbb{C}[\Lambda_T]^{W(\mathbf{G},\mathbf{S})}\right).$$

Using the twisted Satake isomorphism of (see fo example [Bou21a, Theorem 5.2.1]) we obtain

$$\widehat{\mathbf{S}}/W(\mathbf{G}, \mathbf{S}) = \mathrm{Spec}\left(\mathcal{H}_K(\mathbb{C})\right). \tag{1}$$

3. Unramified representations and unramified L-parameters

Let $\mathcal{W}_F \subset \Gamma_{un}$ whose elements induce an integral power of the Frobenius automorphism $\sigma\colon x \mapsto x^q$ on the algebraic closure of the residue field. The valuation $\mathrm{val}\colon \mathcal{W}_F \to \mathbb{Z}$ sends an element $\psi \in \mathcal{W}_F$ to the power of σ it induces, e.g $\mathrm{val}(\sigma) = 1$. Define the "Weyl form" of the Langlands group to be ${}^L_w\mathbf{G} := \widehat{\mathbf{G}} \rtimes \mathcal{W}_F \subset {}^L\mathbf{G}$. The isomorphism $\mathbb{Z} \to \mathcal{W}_F$ given by $1 \mapsto \sigma$ defines a semidirect product $\widehat{\mathbf{G}} \rtimes \mathbb{Z}$ and we get a homomorphism

$${}^L_w\mathbf{G} \to \widehat{\mathbf{G}} \rtimes \mathbb{Z}.$$

Definition 3.1. An unramified L-parameter is a homomorphism $\phi\colon \mathcal{W}_F \to {}^L_w\mathbf{G}$ that verifies the following properties:

(i) The composition $\mathcal{W}_F \xrightarrow{\phi} {}^L_w\mathbf{G} \twoheadrightarrow \mathcal{W}_F$ is the identity.
(ii) For any $w \in \mathcal{W}_F$, $\phi(w)$ is semisimple.
(iii) The composition $\mathcal{W}_F \xrightarrow{\phi} {}^L_w\mathbf{G} \longrightarrow \widehat{\mathbf{G}} \rtimes \mathbb{Z}$ factors through val.

Set $\Phi_{un}(\mathbf{G})$ for the set of equivalence[4] classes of unramified L-parameters.

The set of L-parameters is in bijection with the set of semisimple elements of the form $g \rtimes \sigma \in {}^L\mathbf{G}$. Therefore, $\Phi_{un}(\mathbf{G})$ identifies with the set of semisimple elements of $\widehat{\mathbf{G}}$ modulo σ-conjugation.

Definition 3.2. An unramified representation of $\mathbf{G}(F)$ is a homomorphism of groups $\pi\colon \mathbf{G}(F) \to GL(V)$ where V is a $\mathbb{C}$-vector space verifying the following conditions:

(i) π is irreducible.
(ii) The stabilizer of any vector $v \in V$ is an open subgroups of $\mathbf{G}(F)$.
(iii) For any open subgroup $O \subset \mathbf{G}(F)$, the vector subspace V^O of O-fixed vectors is finite dimensional.
(iv) The subspace V^K is nonzero.

Set $\Pi_{un}(\mathbf{G})$ for the set of equivalence[5] classes of unramified representations of $\mathbf{G}(F)$.

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[4] Two L-parameters are equivalent if they are $\widehat{\mathbf{G}}(\mathbb{C})$-conjugate.

[5] Two representations (π_1, V_1) and (π_2, V_2) are equivalent if there exists an isomorphism $V_1 \to V_2$ sending π_1 to π_2.

Proposition 3.3. *There is a natural bijection*

$$\Phi_{un}(\mathbf{G}) \simeq \widehat{\mathbf{S}}(\mathbb{C})/W(\mathbf{G},\mathbf{S}) \simeq \Pi_{un}(\mathbf{G}).$$

Proof. In the proof of [BR94, Proposition 1.12.1], one shows first the above proposition for the torus $\mathbf{T}$:

$$\Phi_{un}(\mathbf{T}) \simeq \widehat{\mathbf{S}}(\mathbb{C}) \simeq \Pi_{un}(\mathbf{T}),$$

then deduce it for $\mathbf{G}$ using [Bor79, Proposition 6.7]. □

Combining Proposition 3.3 and (1) yields

$$\Phi_{un}(\mathbf{G}) \simeq \mathrm{Spec}(\mathcal{H}_K(\mathbb{C})). \tag{2}$$

Remark 3.4. The above proposition gives an alternative characterization of the untwisted Satake homomorphism. Consider the following injective homormophism

$$\begin{tikzcd} \mathcal{H}_K(\mathbb{C}) \arrow[r] & \{\Pi_{un}(\mathbf{G}) \to \mathbb{C}\} \\ h_g = \mathbf{1}_{KgK} \arrow[r, mapsto] & (\pi \mapsto Tr(\pi(h_g)|_{V^K})), \end{tikzcd}$$

where, V is given a structure of a left $\mathcal{H}_K(\mathbb{C})$-module defined by $f \cdot v$ for $f \in \mathcal{H}_K(\mathbb{C})$ and $v \in V$ by the formula

$$f \cdot v = \int_G f(g)(\pi(g) \cdot v) d\mu_K(g).$$

By Proposition 3.3 we get the following commutative diagram

$$\begin{tikzcd} \mathcal{H}_K(\mathbb{C}) \arrow[rr, hook, "\mathcal{S}_T^G"] \arrow[d, "\simeq"] & & \mathcal{C}_c(\Lambda_T, \mathbb{C}) \arrow[d, "\simeq"] \\ \mathbb{C}[\Pi_{un}(\mathbf{G})] \arrow[r, "\simeq"] & \mathbb{C}[\Pi_{un}(\mathbf{T})]^{W(\mathbf{G},\mathbf{S})} \arrow[r, hook] & \mathbb{C}[\Pi_{un}(\mathbf{T})]. \end{tikzcd}$$

4. The Hecke polynomial

Let $\mathfrak{c} \in \mathcal{M}(\overline{F})$ and $\mu_{\mathfrak{c}} \in X_*(\mathbf{T})$ be the unique $\mathbf{B}_{\overline{F}}$-dominant cocharacter of $\mathbf{T}_{\overline{F}}$. Both, $\mathfrak{c}$ and $\mu_{\mathfrak{c}}$ have the same field of definition, a finite unramified extension $F(\mathfrak{c}) \subset F^{un}$ of F. Set $d = [F(\mathfrak{c}) : F]$ and let

$$Norm_{F(\mathfrak{c})/F}\mathfrak{c} := [\prod_{\tau \in \mathrm{Gal}(F(\mathfrak{c})/F)} \tau(\mu_{\mathfrak{c}})] \in \mathcal{M}(F)$$

be the norm of $\mathfrak{c}$[6]. We may assume that for some representative of the conjugacy class $Norm_{F(\mathfrak{c})/F}\mathfrak{c}$ takes values in the torus $\mathbf{T}$ (and hence for all). The conjugacy class $\mathfrak{c} \in \mathcal{M}(F(\mathfrak{c}))$ determines a Weyl orbit of a character of $\widehat{\mathbf{T}}$, in which there is a unique $\widehat{\mu}_{\mathfrak{c}} \in X^*(\widehat{\mathbf{T}})$ that is dominant with respect to the Borel subgroup $\widehat{\mathbf{B}}$.

Let $(r_{\mathfrak{c}}, V)$ be a representation of ${}^L(\mathbf{G}_{F(\mathfrak{c})})$ (unique up to isomorphism) satisfying the conditions:

- The restriction of $r_{\mathfrak{c}}$ to $\widehat{\mathbf{G}}$ is irreducible with highest weight $\widehat{\mu}_{\mathfrak{c}}$.

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[6] It is straightforward that the conjugacy class $Norm_{F(\mathfrak{c})/F}\mathfrak{c}$ does not depend on the choice of the representative $\mu_{\mathfrak{c}}$.

- For any admissible invariant splitting of ${}^L(\mathbf{G}_{F(\mathfrak{c})})$ the subgroup Γ^d_{un} of ${}^L(\mathbf{G}_{F(\mathfrak{c})})$ acts trivially on the highest weight space of $r_{\mathfrak{c}}$.

Fix an invariant admissible splitting ${}^L(\mathbf{G}_{F(\mathfrak{c})}) = \widehat{\mathbf{G}} \rtimes \Gamma^d_{un}$.

Definition 4.1 The Hecke polynomial. For every $\widehat{g} \in \widehat{\mathbf{G}}$, consider the following polynomial:

$$P_{\mathbf{G},\mathfrak{c}}(X) = \det\left(X - q^{d\langle \mu_{\mathfrak{c}}, \rho\rangle} r_{\mathfrak{c}}\left((\widehat{g} \rtimes \sigma)^d\right)\right).$$

By varying $\widehat{g}$, the coefficients of $P_{\mathbf{G},\mathfrak{c}}$ are viewed as elements of the algebra of regular functions of $\Phi_{un}(\mathbf{G})$. Let $H_{\mathbf{G},\mathfrak{c}} \in \mathcal{H}_K(\mathbb{C})[X]$ be the Hecke polynomial corresponding to $P_{\mathbf{G},\mathfrak{c}}$ via (2) (compare with [BR94, §6]).

5. Explicit twisted Satake transform

Let $\mu \in Norm_{F(\mathfrak{c})/F}\mathfrak{c}$ be the cocharacter of $\mathbf{T}$ which is $\mathbf{B}$-dominant, i.e. ϖ^μ is antidominant. Let $\mathbf{L}$ be the centralizer of μ in $\mathbf{G}$. Let $\mathbf{P}$ be the largest parabolic subgroup of $\mathbf{G}$ relative to which μ is dominant, $\mathbf{L}$ is a Levi factor of $\mathbf{P}$ and $\mathbf{U}^+_P$ the unipotent radical of $\mathbf{P}$. By definition we have $\mathbf{T} \subset \mathbf{L}$ and $\mathbf{U}^+_P \subset \mathbf{U}^+$. Set $K_\dagger = \dagger \cap K$ for any $\dagger \in \{P, L, U^+_P\}$. Set $f_{[\mu]} = \mathbf{1}_{K\varpi^\mu K} \in \mathcal{H}_K(\mathbb{Z})$, $g_{[\mu]} = \mathbf{1}_{\varpi^\mu K_L} \in \mathcal{C}_c(L /\!\!/ K_L, \mathbb{Z})$ and $i_{\varpi^\mu} = \mathbf{1}_{I\varpi^\mu I} \in \mathcal{H}_I(\mathbb{Z})$. Let $p\colon \mathbf{G}_{sc} \to \mathbf{G}$ be the simply connected covering of the derived group of G and let $\mathbf{S}_{sc}$ be the unique maximal F-split torus of $\mathbf{G}_{sc}$ such that $p(\mathbf{S}_{sc}) \subset \mathbf{S}$. The map p defines a homomorphism from $X_*(\mathbf{S}_{sc})$ to $X_*(\mathbf{S})$. We are interested in the set

$$\Sigma_F(\mu) = \{\nu \in X_*(\mathbf{S}) : \mu - \nu \in \mathrm{Im}(X_*(\mathbf{S}_{sc})) \text{ and } w\nu \preceq \mu \text{ for all } w \in W(\mathbf{G}, \mathbf{S})\}.$$

Remark 5.1. The above W-invariant sets of weights plays a prominent role in representation theory and they are called "saturated sets of weights". Moreover, we have (see [Kot84a, §2.3], [Hum72, 13.4 Exercise] and Bourbaki's [Bou68, Chapter VI, Exercises of §1 and §2]) that

$$\Sigma_F(\mu) = \bigsqcup_{\lambda \in X_*(\mathbf{S}) \cap \overline{\mathcal{C}}\colon\, \lambda \preceq \mu} W\lambda$$

where $\preceq$ denotes[7] the partial order on $X_*(\mathbf{S}) \cap \overline{\mathcal{C}}$ defined by

$$\lambda \preceq \nu \Leftrightarrow \nu - \lambda = \sum n_\alpha \alpha^\vee, n_\alpha \in \mathbb{Z}_{\geqslant 0}.$$

Moreover, when μ is minuscule then $\Sigma_F(\mu) = W(\mathbf{G}, \mathbf{S})\mu$ [Bou21a, Remark 5.2.8].

We have the following explicit description of the twisted Satake homomorphism

Proposition 5.2. *Write*

$$\dot{\mathcal{S}}^G_T(f_{[\mu]}) = \sum_{\nu \in \Sigma_F(\mu)} c(\nu).\mathbf{1}_{\varpi^\nu T_1} \in \mathcal{C}_c(T /\!\!/ T_1, \mathbb{Z}),$$

and the coefficients $\{c(\nu)\}$ are positive powers of q and verifies

$$c(w\nu) = q^{\langle \delta, \nu - w(\nu)\rangle} c(\nu) \text{ for all } w \in W(\mathbf{G}, \mathbf{S}), \text{ with } c(\mu) = 1.$$

Proof. This is a particular case of [Bou21a, Theorem 5.2.1 & Theorem 5.3.1]. The twisted Satake isomorphism ensures that $\dot{\mathcal{S}}^G_T(f_{[\mu]}) \in \mathcal{C}_c(T /\!\!/ T_1, \mathbb{Z})^{\dot{W}}$ where $\dot{W}$ denotes the Weyl group with its

---

[7] Compare with [Bou21a, Definition 5.2.4]

twisted dot-action (See [Bou21a, §3.16]). This shows that

$$c(\nu)q^{\langle\delta,\nu\rangle} = c(w(\mu))q^{\langle\delta,w(\nu)\rangle} \text{ for all } w \in W(\mathbf{G},\mathbf{S}).$$

The coefficient $c(\mu) = 1$ is obtained by [Kot84a, Lemma 2.3.7 (b)] using [Bou21a, Remark 5.2.9].

The fact that $c(\nu) > 0$ if and only $\nu \in \Sigma_F(\mu)$ is well known in this unramified case; it follows by [Kot84a, Lemma 2.3.7 (a)] for the "only if" and [Rap00] for the "if". □

6. Seed relations and $\mathbb{U}$-operators

Using the fixed épinglage, we can consider a Γ_{un} -equivariant embedding ${}^L\mathbf{T} = \widehat{\mathbf{T}} \rtimes \Gamma_{un} \hookrightarrow {}^L\mathbf{G}$. The composition

$$ {}^L(\mathbf{T}_{F(\mathfrak{c})}) \longhookrightarrow {}^L(\mathbf{G}_{F(\mathfrak{c})}) \xhookrightarrow{r_{\mathfrak{c}}} GL(V) \xrightarrow{P_{\mathbf{G},\mathfrak{c}}} \mathbb{C}[X] \ ,$$

is independent of all fixed choices. The restriction of $r_{\mathfrak{c}}$ to $\widehat{\mathbf{T}}$ yields a weight space decomposition

$$V = \bigoplus_{\lambda\in\Sigma_E(\mu_{\mathfrak{c}})} V_{\widehat{\lambda}}.$$

We have

$$\mathcal{S}_T^G(P_{\mathbf{G},\mathfrak{c}}) = \det\left(X - q^{d\langle\mu_{\mathfrak{c}},\rho\rangle} r_{\mathfrak{c}}|_{{}^L(\mathbf{T}_{F(\mathfrak{c})})}\left((\widehat{t} \rtimes \sigma)^d\right)\right) \in \mathbb{C}[\Phi_{un}(\mathbf{T})]^{W(\mathbf{G},\mathbf{S})}.$$

Define the twisted restriction of $r_{\mathfrak{c}}$ to be the morphism of schemes

$$r_T \colon {}^L(\mathbf{T}_{F(\mathfrak{c})}) = \widehat{\mathbf{T}} \rtimes \Gamma_{un}^d \longrightarrow GL(V)$$

given on $\mathbb{C}$-points by

$$r_T(1 \rtimes \sigma^d) = r_{\mathfrak{c}}(1 \rtimes \sigma^d) \text{ and } r_T(\widehat{t} \rtimes 1)\cdot v_\lambda = q^{-\langle\rho,\lambda\rangle}\lambda(\widehat{t})\cdot v_\lambda \tag{3}$$

for $v_\lambda \in V_\lambda$ for all $\lambda \in \Sigma(\mu_{\mathfrak{c}})$. The homomorphism r_T is not a homomorphism of groups but maps conjugacy classes to conjugacy classes and it is defined to ensure, using [Bou21a, Remark 5.2.9] and (3), that

$$\begin{aligned}\dot{\mathcal{S}}_T^G(P_{\mathbf{G},\mathfrak{c}}) &= \eta_B \circ \mathcal{S}_T^G(P_{\mathbf{G},\mathfrak{c}}) \\ &= \det\left(X - q^{-d\langle\mu_{\mathfrak{c}},\rho\rangle} r_T\left((\widehat{t} \rtimes \sigma)^d\right)\right) \in \mathbb{C}[\Phi_{un}(\mathbf{T})].\end{aligned}$$

Remark 6.1. Note that our choice of the twisted representation r_T depends crucially on the normalization of the isomorphism $X_*(\mathbf{S}) \simeq \Lambda_T$. We have adopted the following isomorphism $\lambda \mapsto \varpi^\lambda$. Using [Bou21a, Remark 5.2.9] and $\delta_B(\varpi^\lambda)^{1/2} = q^{-\langle\lambda,\rho\rangle}$, we see that

$$\begin{tikzcd}
X_*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{C} \arrow[r, "\eta\colon \lambda\mapsto q^{-\langle\lambda,\rho\rangle}"] \arrow[d, "\simeq"', "\lambda\mapsto\varpi^\lambda T_1"] & X_*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{C} \arrow[d, "\simeq"] \\
\Lambda_T \otimes_{\mathbb{Z}} \mathbb{C} \arrow[r, "\eta\colon tT_1\mapsto\delta(t)^{1/2}tT_1"] & \Lambda_T \otimes_{\mathbb{Z}} \mathbb{C}.
\end{tikzcd}$$

As opposed to [Wed00, Proposition 2.7], we insist on the fact that we do not assume μ to be minuscule in the following proposition.

Proposition 6.2. *(i) Let $\mathbf{S}^{F(\mathfrak{c})} \subset \mathbf{T}$ denotes the maximal split torus of $\mathbf{G}_{F(\mathfrak{c})}$ containing the image of $\mu_{\mathfrak{c}}$, let $\overline{\mathcal{C}}_{F(\mathfrak{c})} \subset \mathcal{B}(\mathbf{G}_{F(\mathfrak{c})}, F(\mathfrak{c}))_{ext}$ be the closed vectorial chamber corresponding to* *the Borel* $\mathbf{B}_{F(\mathfrak{c})}$. *We have*

$$deg(H_{\mathbf{G},\mathfrak{c}}) \geqslant \sum_{\lambda\in X_*(\mathbf{S}^{F(\mathfrak{c})})\cap\overline{\mathcal{C}}_{F(\mathfrak{c})}\colon \lambda\preceq\mu_{\mathfrak{c}}} \#(W(\mathbf{G},\mathbf{S}^{F(\mathfrak{c})})\lambda) = \#\Sigma_{F(\mathfrak{c})}(\mu_{\mathfrak{c}})$$

(ii) *The twisted restriction* r_T *of* $r_{\mathfrak{c}}$ *to* ${}^L(\mathbf{T}_{F(\mathfrak{c})})$ *is isomorphic to a direct sum*

$$V = \bigoplus_{\Sigma_{F(\mathfrak{c})}(\mu_{\mathfrak{c}})} V_{\widehat{\lambda}}$$

where, $V_{w(\widehat{\mu})}$ *is one-dimensional with generator* $v_{\widehat{\lambda}}$ *for any* $w\in W$*, such that*

$$r_T(\widehat{t}\rtimes\sigma^d)\cdot v_{\sigma^{d(r-1)}w(\widehat{\mu})} = q^{-\langle\rho,w(\mu)\rangle}w(\widehat{\mu})(\widehat{t})\cdot v_{w(\widehat{\mu})}. \tag{4}$$

Proof. We will just imitate the proof of [Wed00, (2) Proposition 2.7] but without requiring μ to be minuscule.

(i) Fix a Borel pair $(\widehat{\mathbf{T}},\widehat{\mathbf{B}})$ of $\widehat{\mathbf{G}}$ and let $\widehat{\mu}_{\mathfrak{c}}$ be the dominant character of $\widehat{\mathbf{T}}$ corresponding to the conjugacy classe $\mathfrak{c}$. By definition of the Hecke polynomial, its degree is the dimension of the representation $r_{\mathfrak{c}}$ which is irreducible with highest weight $\widehat{\mu}_{\mathfrak{c}}$ as a representation of $\widehat{\mathbf{G}}$. By remark 5.1, the only weights of $r_{\mathfrak{c}}$ are the elements $\bigsqcup_{\widehat{\lambda}} W(\widehat{\mathbf{G}},\widehat{\mathbf{T}})\widehat{\lambda}$ where the disjoint union is taken over dominant wights $\widehat{\lambda}\preceq\widehat{\mu}_{\mathfrak{c}}$ (here $\preceq$ is the usual partial order on dominant weights $X^*(\widehat{\mathbf{T}})^{\mathrm{dom}}$). By definition of the dual group, we then have

$$\begin{aligned}\bigsqcup_{\widehat{\lambda}\in X^*(\widehat{\mathbf{T}})^{\mathrm{dom}}\colon \widehat{\lambda}\preceq\widehat{\mu}_{\mathfrak{c}}} W(\widehat{\mathbf{G}},\widehat{\mathbf{T}})\widehat{\lambda} &= \bigsqcup_{\lambda\in X_*(\mathbf{S}^{F(\mathfrak{c})})\cap\overline{\mathcal{C}}_{F(\mathfrak{c})}\colon \lambda\preceq\mu_{\mathfrak{c}}} W(\mathbf{G}_{F(\mathfrak{c})},\mathbf{S}^{F(\mathfrak{c})})\lambda\\ &= \Sigma_{F(\mathfrak{c})}(\mu_{\mathfrak{c}}).\end{aligned}$$

(ii) The twisted restriction r_T of $r_{\mathfrak{c}}$ to ${}^L(\mathbf{T}_{F(\mathfrak{c})})$ is isomorphic to a direct sum

$$V = \bigsqcup_{\widehat{\lambda}\in X^*(\widehat{\mathbf{T}})^{\mathrm{dom}}\colon \widehat{\lambda}\preceq\widehat{\mu}_{\mathfrak{c}}} V_{\widehat{\lambda}}$$

and the highest weight space $V_{\widehat{\mu}_{\mathfrak{c}}}$ is one-dimensional[8] with generator $v_{\widehat{\mu}_{\mathfrak{c}}}$. Accordingly, $V_{\widehat{\lambda}}$ is one-dimensional for any $\widehat{\lambda}\in W(\widehat{\mathbf{G}},\widehat{\mathbf{T}})\widehat{\mu}_{\mathfrak{c}}$. The conjugacy class $\mathfrak{c}$ being defined over $F(\mathfrak{c})$, we see that $\langle\sigma^n\rangle$ stabilizes $W(\widehat{\mathbf{G}},\widehat{\mathbf{T}})\widehat{\mu}_{\mathfrak{c}}$.
Choose for each classe $Z\in W(\widehat{\mathbf{G}},\widehat{\mathbf{T}})\widehat{\mu}_{\mathfrak{c}}/\langle\sigma^d\rangle$ a representative $\widehat{\lambda}_Z\in Z$ and a vector $v_{\widehat{\lambda}_Z}\in V_{\widehat{\lambda}_Z}$. Define

$$v_{\sigma^{rd}(\widehat{\lambda}_Z)} := r_{\mathfrak{c}}(1\rtimes\sigma^{rd})\cdot v_{\widehat{\lambda}_Z}, \quad \text{for } 1\leqslant r< r_Z := \min\{s\colon \sigma^{sd}\widehat{\lambda}_Z=\widehat{\lambda}_Z\}.$$

Therefore, taking $r=-1$ gives

$$r_T(\widehat{t}\rtimes\sigma^d)\cdot v_{\sigma^{d(r-1)}(\widehat{\lambda}_Z)} = r_T(\widehat{t}\rtimes 1)\cdot v_{\widehat{\lambda}_Z} \overset{(3)}{=} q^{-\langle\rho,\lambda\rangle}\widehat{\lambda}_Z(\widehat{t})\cdot v_{\widehat{\lambda}_Z}.\square$$

Lemma 6.3. *We have* $(\dot{\mathcal{S}}^G_T H_{\mathbf{G},\mathfrak{c}})(\mu)=0$ *in* $\mathcal{C}_c(T\,/\!\!/\,T_1,R)$.

Proof. The conjugacy classe $[\mu]$ (resp. $\mathfrak{c}$) gave rise to a dominant character $\widehat{\mu}$ (resp. $\widehat{\mu}_{\mathfrak{c}}$) of $\widehat{\mathbf{T}}$ and

$$\widehat{\mu} = \widehat{\mu}_{\mathfrak{c}}\sigma(\widehat{\mu}_{\mathfrak{c}})\cdots\sigma^{d-1}(\widehat{\mu}_{\mathfrak{c}}).$$

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[8] The weight spaces in the weyl orbit of the highest weight are one dimensional, but outside this distinguished weyl orbit, there are weight spaces which are not 1 dimensional.

To prove the lemma, it suffices to show that

$$\det\left(X - q^{d\langle\mu_{\mathfrak{c}},\rho\rangle} r_T|_{V_{\widehat{\mu}_{\mathfrak{c}}}}\left((\sigma \ltimes \widehat{t})^d\right)\right) \in \mathbb{C}[\Phi_{un}(\mathbf{T})][X]$$

has $\widehat{\mu}(\widehat{t})$ as a root for all $\widehat{t} \in \widehat{\mathbf{T}}$. Identify $\Phi_{un}(\mathbf{T})$ with the set of σ-conjugacy classes $\{\widehat{t}\}$ of elements $\widehat{t} \in \widehat{\mathbf{T}}(\mathbb{C})$. For any $v \in V_{\widehat{\mu}}$, we have

$$\begin{aligned}
q^{d\langle\mu_{\mathfrak{c}},\rho\rangle} r_T\left((\sigma \ltimes \widehat{t})^d\right)\cdot v &= q^{d\langle\mu_{\mathfrak{c}},\rho\rangle} r_T\left(\sigma^d \ltimes (\widehat{t}\sigma(\widehat{t})\cdots\sigma^{d-1}(\widehat{t}))\right)\cdot v \\
&\overset{\text{Prop. 6.2}}{=} \widehat{\mu}_{\mathfrak{c}}\left(\widehat{t}\sigma(\widehat{t})\cdots\sigma^{d-1}(\widehat{t})\right)\cdot v \\
&= \widehat{\mu}_{\mathfrak{c}}(\widehat{t})\sigma(\widehat{\mu}_{\mathfrak{c}})(\widehat{t})\cdots\sigma^{d-1}(\widehat{\mu}_{\mathfrak{c}})(\widehat{t})\cdot v \\
&= \widehat{\mu}(\widehat{t})\cdot v. \square
\end{aligned}$$

We will show now the main theorem of the paper:

THEOREM 6.4 Seed relation. *The operator* $u_{\varpi^\mu} \in \mathbb{U}$ *is a right root of the Hecke polynomial* $H_{\mathbf{G},\mathfrak{c}}$ *in the non-commutatif* R*-algebra* $\mathrm{End}_P(\mathcal{C}_c(G/K,R))$.

Proof. Under the identifications $\Lambda_T \simeq X_*(\mathbf{T})_F \simeq X^*(\widehat{\mathbf{T}})_F$ the element $\varpi^\mu T_1 \in \Lambda_T^-$ corresponds to the function $t \mapsto \widehat{\mu}(t)$. Recall that by [Bou21d, Lemma 2.6.4] $u_{\varpi^\mu} \in End_P\mathcal{C}_c(G /\!/ K, \mathbb{Z})$ and the coefficients of $H_{\mathbf{G},\mathfrak{c}}$ are in $\mathcal{H}_K(R) \simeq End_G\mathcal{C}_c(G /\!/ K, R)$ [Wed00, 2.8], thus

$$H_{\mathbf{G},\mathfrak{c}}(u_{\varpi^\mu}) \in End_P\mathcal{C}_c(G /\!/ K, R).$$

Thanks to the compatibility of the Satake and Bernstein twisted isomorphisms [Bou21a, Theorem 6.5.1], we see that $\dot{\Theta}_{\mathrm{Bern}} \circ \dot{\mathcal{S}}_T^G(H_{\mathbf{G},\mathfrak{c}}) \in Z(\mathcal{H}_I(R))[X]$. Write $H_{\mathbf{G},\mathfrak{c}} = \sum_{k=1}^r h_k X^k$ and $\bar{h}_k = \dot{\Theta}_{\mathrm{Bern}} \circ \dot{\mathcal{S}}_T^G(h_k) \in Z(\mathcal{H}_I(R))$. So $\bar{h}_k *_I \mathbf{1}_K = \mathbf{1}_K *_I \bar{h}_k = h_K$. We then have for any $p \in P$

$$\begin{aligned}
\mathbf{1}_{pK} \bullet H_{\mathbf{G},\mathfrak{c}}(u_{\varpi^\mu}) &= \sum_{k=1}^r (\mathbf{1}_{pK} \bullet u_{\varpi^\mu}^k) *_K h_k \\
&= \sum_{k=1}^r (\mathbf{1}_{pI} *_I i_{\varpi^\mu}^k) *_I \mathbf{1}_K *_K h_k \\
&= \sum_{k=1}^r (\mathbf{1}_{pI} *_I i_{\varpi^\mu}^k) *_I (\frac{1}{[K:I]}\mathbf{1}_K *_I \mathbf{1}_K *_I \bar{h}_k) \\
&= \sum_{k=1}^r (\mathbf{1}_{pI} *_I i_{\varpi^\mu}^k) *_I \mathbf{1}_K *_I \bar{h}_k \\
&= \mathbf{1}_{pI} *_I \left(\sum_{k=1}^r i_{\varpi^\mu}^k *_I \bar{h}_k\right) *_I \mathbf{1}_K \\
&= \mathbf{1}_{pI} *_I \left(\sum_{k=1}^r \bar{h}_k *_I i_{\varpi^\mu}^k\right) *_I \mathbf{1}_K \\
&= \mathbf{1}_{pI} *_I \left((\dot{\Theta}_{\mathrm{Bern}} \circ \dot{\mathcal{S}}_T^G H_{\mathbf{G},\mathfrak{c}})(i_{\varpi^\mu}^k)\right) *_I \mathbf{1}_K \\
&= \mathbf{1}_{pI} *_I \dot{\Theta}_{\mathrm{Bern}}\left((\dot{\mathcal{S}}_T^G H_{\mathbf{G},\mathfrak{c}})(\varpi^\mu T_1)\right) *_I \mathbf{1}_K \\
&\overset{\text{Lemma 6.3}}{=} 0.
\end{aligned}$$

We have shown $H_{\mathbf{G},\mathfrak{c}}(u_{\varpi^\mu}) = \sum_{k=1}^r h_k \circ u_{\varpi^\mu}^k = 0 \in End_P(\mathcal{C}_c(G/K,R))$. $\square$

Remark 6.5. If $\mu_{\mathfrak{c}}$ is minuscule, then $\Sigma_F(\mu_{\mathfrak{c}}) = W(\mathbf{G}_{\overline{F}}, \mathbf{T})\,\mu_{\mathfrak{c}}$ and accordingly the degree of the Hecke polynomial is

$$deg(H_{\mathbf{G},\mathfrak{c}}) = \left|W(\mathbf{G}_{\overline{F}}, \mathbf{T})\,\mu_{\mathfrak{c}}\right|.$$

In particular, $deg(H_{\mathbf{G},\mathfrak{c}}) \geqslant deg(P_\mu) = |W/W_\mu| = |W(\mathbf{G}, \mathbf{S})\,\mu|$, where P_μ is the minimal polynomial of u_{ϖ^μ} in $Z(\mathcal{H}_I(R))$ (see proof of [Bou21d, Theorem 2.8.1]). Therefore, if $\mathbf{G}$ is a split group, $\mu_{\mathfrak{c}}$ minuscule and $E = F$, then

$$H_{G,[\mu]} = P_\mu *_I \mathbf{1}_K.$$

7. Bültel's annihilation relation

In this last section we will show how Theorem 6.4 lifts (generalizes) a previously known result due to Bültel [Bül97, 1.2.11].

Let $\dot{\mathcal{S}}_P \colon \mathcal{C}_c(P/K_P, \mathbb{Q}) \to \mathcal{C}_c(L/K_L, \mathbb{Q})$ be the canonical homomorphism given by

$$f \mapsto \left(m \mapsto \int_{U_P^+} f(nm)d\mu_{U_P^+}(n)\right),$$

where $d\mu_{U_P^+}$ is the left-invariant Haar measure giving $K_{U_P^+}$ volume 1. Both $\mathbb{Q}$-modules $\mathcal{C}_c(P/K_P, \mathbb{Q})$ and $\mathcal{C}_c(L/K_L, \mathbb{Q})$ are actually $\mathbb{Q}$-algebras (by [Bou21a, Lemma 3.11.2]) and the transform $\dot{\mathcal{S}}_P$ is an algebra homomorphism. Indeed, let $f, g \in \mathcal{C}_c(P/K_P, \mathbb{Q})$ then

$$\begin{aligned}
\dot{\mathcal{S}}_P(f *_{K_P} g)(p) &= \int_{U_P^+} (\int_P f(a)g(a^{-1}up)d\mu_P(a))d\mu_{U_P^+}(u)\\
&= \int_{U_P^+}\int_L\int_{U_P^+} f(nm)g(m^{-1}n^{-1}up)d\mu_{U_P^+}(n)d\mu_L(m)d\mu_{U_P^+}(u)\\
&= \int_{U_P^+}\left(\int_L f(nm)d\mu_{U_P^+}(n)\right)\left(\int_{U_P^+} g(m^{-1}pu)d\mu_{U_P^+}(u)\right)d\mu_L(m)\\
&= \dot{\mathcal{S}}_P(f) *_{K_P} \dot{\mathcal{S}}_P(g)(p)
\end{aligned}$$

where, $d\mu_P$ denotes the left invariant Haar measure giving K_P measure 1.

We also consider the map $|_P$ sending any function on G to its restriction to P. Using the Iwasawa decomposition $G = PK$ ([Bou21a, Proposition 2.2.1]) one shows that this is actually an algebra homomorphism

$$|_P \colon \mathcal{H}_K(R) \longrightarrow \mathcal{C}_c(P /\!\!/ K_P, R),$$

and a $|_P$-linear module homomorphism

$$|_P \colon \mathcal{C}_c(G/K, R) \longrightarrow \mathcal{C}_c(P/K_P, R).$$

Lemma 7.1. *Let $p \in P$ and $m \in L$, then:*

$$\mathbf{1}_{pK}|_P = \mathbf{1}_{pK_P} \text{ and } \dot{\mathcal{S}}_L^P(\mathbf{1}_{mK_P}) = |mK_{U_P^+}m^{-1}|_{U_P^+}\mathbf{1}_{mK_L}.$$

Proof. The first equality is a direct consequence of the Iwasawa decomposition. For the second it is deduced from the fact that $K_P = K_L.K_{U_P^+}$ given in [Bou21a, Proposition 2.2.1]:

$$\dot{\mathcal{S}}_L^P(\mathbf{1}_{mK_P})(a) = \int_{U_P^+} \mathbf{1}_{mK_P}(ua)d\mu_{U_P^+}(u).$$

The integrand is nonzero if and only if $ua \in mK_P = mK_L \cdot K_{U_P^+}$, but since $L \cap U_P^+ = \{1\}$, we have

$$u \in aK_{U_P^+}a^{-1} \text{ and } a \in mK_L,$$

which is equivalent to $u \in mK_{U_P^+}m^{-1}$ and $w \in mK_L$. Therefore,

$$\dot{\mathcal{S}}_L^P(\mathbf{1}_{mK_P}) = |mK_{U_P^+}m^{-1}|_{U_P^+}\mathbf{1}_{mK_L}.\square$$

Observe that if $mK_{U_P^+}m^{-1} \subset K_{U_P^+}$ then

$$|mK_{U_P^+}m^{-1}|_{U_P^+} = \frac{1}{[K_{U_P^+} : mK_{U_P^+}m^{-1}]} = \frac{1}{[K_P : mK_Pm^{-1}]}.$$

LEMMA 7.2. *We have a following commutative diagram of R-algebras*

$$\begin{tikzcd}
\mathcal{H}_K(R) \arrow[r, hook, "\dot{\mathcal{S}}_L^G"] \arrow[d, "\dot{\mathcal{S}}_T^G"', "\simeq"] & \mathcal{C}_c(L \mathbin{/\mkern-6mu/} K_L, R) \arrow[d, "\simeq"', "\dot{\mathcal{S}}_T^L"] \\
R[\Lambda_T]^{\dot{W}} \arrow[r, hook] & R[\Lambda_T]^{\dot{W}_L}
\end{tikzcd}$$

where, W_L denotes the relative Weyl group of L (which is equal to the subgroup W_μ of elements in W fixing μ). The lowest horizontal arrow is the inclusion of W-invariants into W_L-invariants.

Proof. By definition of the parabolic P, multiplication in G gives a bijection

$$(U^+ \cap L) \cdot U_P^+ \xrightarrow{\sim} U^+. \tag{5}$$

For any $m \in L$ and $h \in \mathcal{H}_K(R)$

$$\begin{aligned}
\dot{\mathcal{S}}_T^G(h)(m) &= \int_{U^+} h(um)d\mu_{U^+}(u) && \text{[Bou21a, Lemma 5.1.2]}\\
&= \int_{U_P^+}\int_{U^+\cap L} h(u_1u_2m)d\mu_{U_P^+}(u_1)\,d\mu_{U^+\cap L}(u_2) && \text{by (5)}\\
&= \int_{U^+\cap L}\left(\int_{U_P^+} h(u_1u_2m)d\mu_{U_P^+}(u_1)\right)d\mu_{U^+\cap L}(u_2)\\
&= \int_{U^+\cap L} \dot{\mathcal{S}}_L^G(h)(u_2m)d\mu_{U^+\cap L}(u_2)\\
&= \dot{\mathcal{S}}_T^L \circ \dot{\mathcal{S}}_L^G(h)(m).
\end{aligned}$$

Therefore, $\dot{\mathcal{S}}_T^G = \dot{\mathcal{S}}_T^L \circ \dot{\mathcal{S}}_L^G$ which confirms the claimed commutativity of the above diagram. Finally, the vertical maps are isomorphisms by [Bou21a, Theorem 5.2.1]. $\square$

Let us reformulate the above twisted Satake homomorphism $\dot{\mathcal{S}}_L^G$ as a homomorphism of endomorphism rings. We have a commutative diagram:

$$\begin{tikzcd}
\mathcal{H}_K(R) \arrow[r, "|_P"] \arrow[d, equal] & \mathcal{C}_c(P \mathbin{/\mkern-6mu/} K_P, R) \arrow[rr, hook, "\dot{\mathcal{S}}_P"] \arrow[d, equal] & & \mathcal{C}_c(L \mathbin{/\mkern-6mu/} K_L, R) \arrow[d, equal] \\
End_G\mathcal{C}_c(G/K, R) \arrow[r, hook, "(1)"] & End_P\mathcal{C}_c(G/K, R) \arrow[r, "(2)"] & End_L\mathcal{C}_c(L/K_L, R) \arrow[r, "(3)", "\simeq"'] & End_L\mathcal{C}_c(L/K_L, R).
\end{tikzcd}$$

Let us first say few words about the homomorphisms (1) and (2):

- We have used the Iwasawa decomposition $G = PK$ to identify $G/K \simeq P/K_P$ for the middle vertical arrow, accordinly the homomorphism $|_P$ induces the canonical injection (1):

$$End_G\mathcal{C}_c(G/K, R) \hookrightarrow End_P\mathcal{C}_c(G/K, R).$$

- We have a homomorphism of rings

$$\begin{array}{ccc} End_P\mathcal{C}_c(G/K, R) & \longrightarrow & End_P\mathcal{C}_c(U_P^+\backslash G/K, R) \\ \\ f & \longmapsto & (U_P^+gK \mapsto \Pi(f(gK))) \end{array}$$

  where Π is the natural obvious map $R[G/K] \to R[U_P^+\backslash G/K]$. But since $P = LU_P^+$, we actually have $End_P\mathcal{C}_c(U_P^+\backslash G/K, R) = End_L\mathcal{C}_c(U_P^+\backslash G/K, R)$.
  Using the Iwasawa decomposition again $G = U_P^+LK$, we get a bijection

$$U_P^+\backslash G/K \simeq L/K_L.$$

  Thus, the homomorphism (2) is the composition

$$End_P\mathcal{C}_c(G/K, R) \longrightarrow End_L\mathcal{C}_c(U_P^+\backslash G/K, R) \xrightarrow{\simeq} End_L\mathcal{C}_c(L/K_L, R).$$

- The homomorphism (3) is the twist by the modulus function δ.

Lemma 7.3. *The operator u_{ϖ^μ} lives in $End_P\mathcal{C}_c(G/K, R)$ and its image by the composition* (3) $\circ$ (2) *is precisely $g_{[\mu]}$.*

Proof. Let us first compute the image of the operator u_{ϖ^μ} by the map (2). We have for all $a \in L$ (see [Bou21d, Lemma 2.6.4])

$$\begin{aligned} u_{\varpi^\mu}(\mathbf{1}_{U_P^+aK}) &= \sum_{p' \in [U_P^+ \cap I^+/U_P^+ \cap \varpi^\mu I^+ \varpi^{-\mu}]} \mathbf{1}_{U_P^+ap'\varpi^\mu K} \\ &= \#(U_P^+ \cap I^+/U_P^+ \cap \varpi^\mu I^+ \varpi^{-\mu})\mathbf{1}_{U_P^+a\varpi^\mu K} \\ &= \#(I^+/\varpi^\mu I^+ \varpi^{-\mu})\mathbf{1}_{U_P^+a\varpi^\mu K} & \text{[Bou21d, Lemma 2.3.2]} \end{aligned}$$

Hence, the image of $u_{\varpi^\mu} \in End_P\mathcal{C}_c(G/K, R)$ by (2) is

$$\#(I^+/\varpi^\mu I^+ \varpi^{-\mu})g_{[\mu]} = \delta_B(\varpi^{-\mu})g_{[\mu]} = q^{2\langle\mu,\rho\rangle}g_{[\mu]}.$$

Finally, (3) shows that the image of u_{ϖ^μ} by the composition (3) $\circ$ (2) is $g_{[\mu]} \in End_L\mathcal{C}_c(L/K_L, R)$. □

Bultel's annihilation result we have mentioned earlier is:

Corollary 7.4 Bultel's annihilation. *We have*

$$\dot{\mathcal{S}}_L^G(H_{\mathbf{G},\mathfrak{c}}(g_{[\mu]})) = 0 \in \mathcal{C}_c(L /\!\!/ K_L, R).$$

Bultel's result as stated in [Wed00, §2.9] requires the conjugacy class $\mathfrak{c}$ to be minuscule. We will derive this corollary from Theorem 6.4, showing that the assumption "minuscule" is superfluous.

Proof. By definition of the "excursion" pairing [Bou21d, §2.6] and the proof of Lemma 7.3, we see that for all $p \in P$:

$$\begin{aligned} 0 &\overset{\text{Theorem 6.4}}{=} (H_{\mathbf{G},\mathfrak{c}}(u_{\varpi^\mu}) \bullet \mathbf{1}_{pK})\,|_P \\ &= \mathbf{1}_{pK_P} *_{K_P} \mathbf{1}_{K_P\varpi^\mu K_P} *_{K_P} (H_{\mathbf{G},\mathfrak{c}})|_P. \end{aligned}$$

This shows that

$$(H_{\mathbf{G},\mathfrak{c}})|_P(\mathbf{1}_{K_P\varpi^\mu K_P}) = 0,$$

and consequently we conclude

$$\dot{\mathcal{S}}_G^L(H_{\mathbf{G},\mathfrak{c}})(g_{[\mu]}) = \dot{\mathcal{S}}_P\left((H_{\mathbf{G},\mathfrak{c}})|_P(\mathbf{1}_{K_P\varpi^\mu K_P})\right) = 0.\square$$

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