GENERAL VERTICAL NORM COMPATIBLE SYSTEMS ON UNITARY SHIMURA VARIETIES

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Abstract. The ultimate goal of this short paper is to construct a vertical norm compatible systems of special Gross–Gan–Prasad cycles, arising from Shimura varieties attached to $U(n-1,1) \hookrightarrow U(n,1) \times U(n-1,1)$.

Contents

A version of this paper has been available since November 2018. In the meantime the paper " " 20? by D. Loeffler has appeared in which the author gives ...

1. Introduction

For this version of the paper, some hyperlinks in blue refer to the paper [Bou21]. For example, Lemma 3.9.5 actually means [Bou21, Lemma 3.9.5].

1.1. **The Gan–Gross–Prasad setting.** Gan, Gross and Prasad formulated some conjectures[1] relating special values of derivatives of automorphic L-functions to heights of certain special cycles on Shimura varieties constructed from embeddings of reductive groups, e.g. [GGP09, Conjecture 27.1]. In this paper, we consider the case of special cycles on higher-dimensional Shimura varieties, where the embedding $\mathrm{Res}_{E/\mathbb{Q}} \mathbf{G}_{m,E} \hookrightarrow \mathrm{GL}_{2,\mathbb{Q}}$ defining Heegner points is replaced by an embedding of unitary groups $\mathbf{U}(n-1,1) \hookrightarrow \mathbf{U}(n,1) \times \mathbf{U}(n-1,1)$ associated to a CM-extension E/F.

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[1]These conjectures may be thought of as Gross–Zagier-type formulas, the cycles being generalizations of classical Heegner points.

Let E be a CM field, that is, an imaginary quadratic extension of a totally real number field F. Set $[E:\mathbb{Q}] = 2[F:\mathbb{Q}] = 2d$. Let τ be the non-trivial element of $\mathrm{Gal}(E/F)$. Fix an integer $n > 1$. Let W be a Hermitian E-space of dimension n and of signature $(n-1,1)$ at one fixed distinguished embedding $\iota\colon E \hookrightarrow \mathbb{C}$ and, of signature $(n,0)$ at the other archimedean places. Let D be a positive definite Hermitian E-line. Consider the $n+1$-dimensional Hermitian E-space $V = W \oplus D$, it has signature $(n,1)$ at the distinguished archimedean place and, signature $(n,0)$ at the other ones.

We consider the F-algebraic reductive groups of unitary isometries $\mathbf{U}(V)$ and $\mathbf{U}(W)$. Set $\mathbf{G}_V := \mathrm{Res}_{F/\mathbb{Q}}\,\mathbf{U}(V)$ and $\mathbf{G}_W := \mathrm{Res}_{F/\mathbb{Q}}\,\mathbf{U}(W)$. We identify $\mathbf{G}_W$ with the subgroup of $\mathbf{G}_V$. Let $\mathbf{G} = \mathbf{G}_V \times \mathbf{G}_W$ and $\mathbf{H} = \Delta(\mathbf{G}_W) \subset \mathbf{G}$, where Δ denotes the diagonal embedding $\Delta\colon \mathbf{G}_W \hookrightarrow \mathbf{G}$.

Let $\mathcal{X}_V$ be the Hermitian symmetric domain consisting of negative definite lines in $V \otimes_{F,\iota} \mathbb{R}$ and similarly let $\mathcal{X}_W$ be the set of negative definite lines in $W \otimes_{F,\iota} \mathbb{R}$. Setting $\mathcal{X} = \mathcal{X}_V \times \mathcal{X}_W$, the diagonal embedding $W \hookrightarrow V \oplus W$ induces an embedding of Hermitian symmetric domains $\mathcal{X}_W$ into $\mathcal{X}$; set $\mathcal{Y}$ for the image of $\mathcal{X}_W$.

The two pairs $(\mathbf{G},\mathcal{X})$ and $(\mathbf{H},\mathcal{Y})$ are Shimura data. For small enough compact open subgroup $K_{\mathbf{G}} \subset \mathbf{G}(\mathbb{A}_f)$ (resp. $K_{\mathbf{H}} \subset \mathbf{G}(\mathbb{A}_f)$), the Shimura variety $\mathrm{Sh}_{K_{\mathbf{G}}}(\mathbf{G},\mathcal{X})$ (resp. $\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H},\mathcal{Y})$) is a complex quasi-projective smooth variety whose $\mathbb{C}$-points are given by

$$\mathbf{G}(\mathbb{Q})\backslash(\mathcal{X} \times (\mathbf{G}(\mathbb{A}_f)/K_{\mathbf{G}})) \quad (\text{resp. } \mathbf{H}(\mathbb{Q})\backslash(\mathcal{Y} \times (\mathbf{H}(\mathbb{A}_f)/K_{\mathbf{H}}))),$$

where $\mathbf{G}(\mathbb{Q})$ (resp. $\mathbf{H}(\mathbb{Q})$) acts diagonally on $\mathcal{X} \times (\mathbf{G}(\mathbb{A}_f)/K_{\mathbf{G}})$ (resp. $\mathcal{Y} \times (\mathbf{H}(\mathbb{A}_f)/K_{\mathbf{H}})$). In fact, these varieties are defined over the reflex field $E = E(\mathbf{G},\mathcal{X}) = E(\mathbf{H},\mathcal{Y})$.

For every $g \in \mathbf{G}(\mathbb{A}_f)$, we will denote by $\mathfrak{z}_g$ the n-codimensional $\mathbf{H}$-special cycle $[\mathcal{Y} \times gK] \subset \mathrm{Sh}_K(\mathbf{G},\mathcal{X})(\mathbb{C})$. Set,

$$\mathcal{Z}_{\mathbf{G},K}(\mathbf{H}) := \{\mathfrak{z}_g : g \in \mathbf{G}(\mathbb{A}_f).\}$$

The natural map $\mathbf{G}(\mathbb{A}_f) \to \mathcal{Z}_{\mathbf{G},K}(\mathbf{H})$ given by $g \mapsto \mathfrak{z}_g$, induces the bijection

$$\mathcal{Z}_{\mathbf{G},K}(\mathbf{H}) \simeq \mathbf{H}(\mathbb{Q})Z_{\mathbf{G}}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}_f)/K,$$

where $Z_{\mathbf{G}} \simeq \mathbf{T}^1$ denotes the center of $\mathbf{G}$. The $\mathbf{H}$-special cycles $\mathcal{Z}_{\mathbf{G},K}(\mathbf{H})$ are all defined over the transfer class field $E(\infty)$.

The Galois group $\mathrm{Gal}(E(\infty)/E)$ acts on the set of special cycles *through* the left action of $H(\mathbb{A}_f)$. More precisely, for every $\sigma \in \mathrm{Gal}(E(\infty)/E)$, we let $h_\sigma \in \mathbf{H}(\mathbb{A}_f)$ be any element verifying $Art_E^1(\det(h_\sigma)\cdot\mathbf{T}^1(\mathbb{Q}) = \sigma|_{E(\infty)}$. For every $g \in \mathbf{G}(\mathbb{A}_f)$, we have

$$\sigma(\mathfrak{z}_g) = \mathfrak{z}_{h_\sigma g}.$$

For $\star \in \{W,V\}$, we fix any compact open subgroups $K_\star \subset \mathbf{U}_\star(\mathbb{A}_{F,f})$. There exists a finite set S of places of F such that $K_\star$ is of the form $K_{\star,S} \times K_\star^S$ where $K_{\star,S}$ is some compact open subgroup of $\mathbf{U}_\star(\mathbb{A}_{F,f}^S)$ and $K_\star^S$ is the product of the hyperspecial compact open subgroups $K_{\star,v} := \underline{\mathbf{U}}_\star(\mathcal{O}_{F_v}) \subset \underline{\mathbf{U}}_\star(F_v)$ for all $v \notin S$. In particular, $K_{W,v} = K_{V,v} \cap \underline{\mathbf{U}}_W(F_v)$. Set $K_v := K_{V,v} \times K_{W,v}$.

1.2. **Main theorem.** Set

$$\mathcal{P}_{sp} := \{\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_F)\colon \mathfrak{p} \text{ is split in } E/F, \mathfrak{p} \notin S, \mathfrak{p} \nmid \mathfrak{c}_1, \mathfrak{p}\mathcal{O}_E \nmid I_0\},$$

where S is defined in [Bou21, §3.6.4], $\mathfrak{c}_1$ in Remark 3.9.2 and I_0 in Lemma 3.9.5 in *loc. cit.*. Denote by $\mathcal{M}_{sp}$ the set of (not necessarily square-free) products of primes in $\mathcal{P}_{sp}$. For every place v in $\mathcal{P}_{sp}$ corresponding to the prime ideal $\mathfrak{p}_v \in \mathcal{N}_{sp}$, let w be the place of E defined by a fixed embedding $\iota_v\colon \overline{F} \hookrightarrow \overline{F}_v$. We denote by $\mathfrak{P}_w$ the prime ideal of $\mathcal{O}_E$ above $\mathfrak{p}_v$ corresponding to the place w and set Fr_w for the corresponding *geometric* Frobenius[2].

[2] Which induces $x \mapsto x^{-q_v}$ on the residue fields of E_v^{un} (resp. E_w^{un} and $E_{\overline{w}}^{un}$).

Let $\xi_1 := \mathfrak{z}_{g_0}$ be the cycle fixed for any g_0 in [Bou21, §3.6.4]. We have fixed in §3.7 *loc. cit.*, a field $\mathcal{K}$ over which the base cycle $\mathfrak{z}_{g_0}$ is defined.

Theorem 1.2.1 (Main theorem)**.** *There exists a collection of cycles $\xi_{\mathfrak{f}} \subset \mathbb{Z}[\mathcal{Z}_{\mathbf{G},K}(\mathbf{H})]$ (for all $\mathfrak{f} \in \mathcal{M}_{sp}$) each defined over $\mathcal{K}(\mathfrak{f})$ (constructed in §4.6.1) such that for every place $v \in \mathcal{P}_{sc}$, with $\mathfrak{p}_v \mid \mathfrak{f}$, we have*

$$\sum_{k=0}^{N} q^{\gamma k} C_{v,k} \operatorname{Tr}_{\mathcal{K}(\mathfrak{p}_v^k \mathfrak{f})/\mathcal{K}(\mathfrak{f})} \xi_{\mathfrak{p}_v^k \mathfrak{f}} = 0$$

where $\gamma = n(n-1)/2$ and $C_{v,k} \in \mathbb{Z}[\mathbf{G}(F_v) /\!\!/ K_v]$ are some fixed Hecke operators in §2.2.

Remark 1.2.2. (i) The methods used for the proof of Theorems 1.3.2 and 1.2.1 for split places give also, *mutatis mutandis*, vertical norm-compatible systems for inert places of F. The details of these calculations for the split vertical case and tame/vertical inert cases will appear in a forthcoming paper, which will also include similar treatment for other embeddings of Shimura data. (ii) Theorem 1.2.1 dates back to the first draft of my thesis in autumn 2018, since then D. Loeffler proposed a general construction for vertical compatible systems which includes this case, and also Cornut's cycles for $\mathbf{SO}(2n+1)$, and many other interesting algebraic-cycle constructions such as cycles on $\mathbf{U}(2n-1,1)$ coming from $\mathbf{U}(n-1,1) \times \mathbf{U}(n,0)$.

2. Local vertical distribution

2.1. **Local notations.** Recall that $\mathbf{B} = \mathbf{B}_1 \times \mathbf{B}_2 \subset \mathbf{G}$ is the Borel subgroup, such that $\mathbf{B}_1 \subset \mathrm{GL}_{n+1}$ (resp. $\mathbf{B}_2 \subset \mathrm{GL}_n$) is the Borel subgroup of upper (resp. lower) triangular matrices. We have denoted by $\mathbf{U}_1$ (resp $\mathbf{U}_2$) the unipotent radical of $\mathbf{B}_1$ (resp. $\mathbf{B}_2$). We also denoted by $\mathbf{T} = \mathbf{T}_1 \times \mathbf{T}_2 \subset \mathbf{B}$ the split maximal torus such that, $\mathbf{T}_1$ (resp. $\mathbf{T}_2$) is the torus of diagonal matrices in GL_{n+1} (resp. in GL_n). Set $\mathbf{U} = \mathbf{U}_1 \times \mathbf{U}_2$, the unipotent radical of $\mathbf{B}$. Consider the following Iwahori subgroup $I = I_1 \times I_2 = \{g \in K : (g \mod \varpi) \in B\}$, i.e.

$$I = \begin{pmatrix} \mathcal{O}_F & \cdots & \cdots & \mathcal{O}_F \\ \varpi\mathcal{O}_F & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \varpi\mathcal{O}_F & \cdots & \varpi\mathcal{O}_F & \mathcal{O}_F \end{pmatrix} \times \begin{pmatrix} \mathcal{O}_F & \varpi\mathcal{O}_F & \cdots & \varpi\mathcal{O}_F \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \varpi\mathcal{O}_F \\ \mathcal{O}_F & \cdots & \cdots & \mathcal{O}_F \end{pmatrix} \subset K_1 \times K_2$$

Let $\mathcal{A}_1$ (resp. $\mathcal{A}_2$) be the apartment corresponding to $\mathbf{T}_1$ (resp. $\mathbf{T}_2$) in the reduced buildings. Recall that the cocharacter group $X_*(\mathbf{T})$ is the free $\mathbb{Z}$-module generated by the basis $\{(\chi_{1,i}, \chi_{2,j}) : (i,j) \in \mathbb{Z}^{n+1} \times \mathbb{Z}^n\}$, where $\chi_{?,i} \in X_*(\mathbf{T}_?)$ $(? \in \{1,2\})$ is the cocharacter mapping x to a diagonal matrix with an x in the (i,i) coordinate and 1 everywhere else. The set of $\mathbf{B}$-dominant cocharacters denoted by T is by definition

$$\{\mu \in X_*(\mathbf{T}) : \mu(\varpi)\mathbf{B}(\mathcal{O}_F)\mu(\varpi)^{-1} \subset \mathbf{B}(\mathcal{O}_F)\}.$$

Using the identification $X_*(\mathbf{T}) \simeq \mathbf{T}(F)/\mathbf{T}(\mathcal{O}_F)$ via the map $\mu \mapsto \mu(\varpi)$ we get:

$$T \simeq \left\{ \left(\operatorname{diag}(\varpi^{a_k})_{1 \le k \le n+1}, \operatorname{diag}(\varpi^{b_k})_{1 \le k \le n}\right) : a_i, b_j \in \mathbb{Z},\ a_1 \ge \cdots \ge a_{n+1},\ b_1 \le \cdots \le b_n \right\}.$$

Let $\eta \in X_*(\mathbf{T}_2)$ given by $t \mapsto \operatorname{diag}(t^n, t^{n-1}, \ldots, t)$ and $\overline{\eta} = (\iota \circ \eta \times \eta^{-1}) \in T$. Fix also the following unipotent element

$$u_0 = \begin{pmatrix} 1 & & 1 \\ & \ddots & \vdots \\ & & 1 \end{pmatrix} \in \mathrm{GL}_{n+1}(F)$$

Let $\mathcal{U}_{\overline{\eta}} \in \operatorname{End}_{\mathbb{Z}[B]} \mathbb{Z}[G/K]$ be the $\mathbb{U}$-operator (see [**?**, §3.3]) associated to the following Iwahori–Hecke operator

$$I\overline{\eta}(\varpi)I = I_1\iota(\eta(\varpi))I_1 \times I_2\eta(\varpi^{-1})I_2 \in \mathcal{H}(G /\!/ I).$$

We will be mainly interested in classes in G/K represented by matrices of the form

$$g_m := (u_0, 1)\overline{\eta}(\varpi^m) = \left(\begin{pmatrix} \varpi^{mn} & & & & 1 \\ & \varpi^{m(n-1)} & & & \vdots \\ & & \ddots & & \vdots \\ & & & \varpi^m & \vdots \\ & & & & 1 \end{pmatrix}, \begin{pmatrix} \varpi^{-mn} & & & \\ & \varpi^{-m(n-1)} & & \\ & & \ddots & \\ & & & \varpi^{-m} \end{pmatrix} \right)$$

Set,

$$v_m = (v_{m,1}, v_{m,2}) := g_m \cdot (v_{0,1}, v_{0,2}) \in \mathcal{A}_{\text{ext}},$$

where $v_0 := (v_{0,1}, v_{0,2})$ is the hyperspecial point fixed by K, such that $v_{0,1} \in \mathcal{A}_{1,\text{ext}}$ (resp. $v_{0,2} \in \mathcal{A}_{2,\text{ext}}$) is the hyperspecial point fixed by K_1 (resp. K_2). Set **Hyp** $:= G \cdot v_0 \simeq G/K$. By definition, for every $k \geq 1$

$$\mathcal{U}_{\overline{\eta}}^k v = \mathcal{U}_{\overline{\eta}^k} v_0 = \sum_{u \in I^+/(I^+ \cap \overline{\eta}(\varpi^k)K\overline{\eta}(\varpi^{-k}))} u\,\overline{\eta}(\varpi^k) \cdot v_0,$$

Where $I^+ = U \cap I = U_1 \cap I_1 \times U_2 \cap I_2$. By B-equivariance (see Corollary 2.6.4 and Lemma 3.3.4), we have for every $m \geq 0$ and $k \geq 1$, $\mathcal{U}_{\overline{\eta}}^k v_m = g_m \mathcal{U}_{\overline{\eta}}^k v_0$.

Lemma 2.1.1. *For every $k \geq 1$, the collection*

$$J'_k := \left\{ (u_{\underline{a},k}, u_{\underline{b},k}) = \left(\begin{pmatrix} 1 & a_{1,2} & \cdots & a_{1,n+1} \\ & \ddots & \ddots & \vdots \\ & & \ddots & a_{n,n+1} \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ b_{2,1} & \ddots & & \\ \vdots & \ddots & \ddots & \\ b_{n,1} & \cdots & b_{n,n-1} & 1 \end{pmatrix} \right) \right\}$$

For all $a_{i,j} \in \mathcal{O}_F/\varpi^{k(j-i)}\mathcal{O}_F$ and $b_{i,j} \in \mathcal{O}_F/\varpi^{k(i-j)}\mathcal{O}_F$, forms a complete set of representatives for $I^+/I^+ \cap \overline{\eta}(\varpi^k)K\overline{\eta}(\varpi^{-k})$.

Proof. This follows from

$$I^+ \cap \overline{\eta}(\varpi^k)K\overline{\eta}(\varpi^{-k}) = \begin{pmatrix} 1 & \varpi^k\mathcal{O}_F & \varpi^{2k}\mathcal{O}_F & \cdots & \varpi^{nk}\mathcal{O}_F \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & \varpi^{2k}\mathcal{O}_F \\ & & & \ddots & \varpi^k\mathcal{O}_F \\ & & & & 1 \end{pmatrix} \times \begin{pmatrix} 1 & & & & \\ \varpi^k\mathcal{O}_F & \ddots & & & \\ \varpi^{2k} & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \\ \varpi^{(n-1)k}\mathcal{O}_F & \cdots & \varpi^{2k}\mathcal{O}_F & \varpi^k\mathcal{O}_F & 1 \end{pmatrix} \subset I^+.$$

□

2.2. **Local vertical compatibility.** Recall the following natural surjective homomorphisms of $\mathbb{Z}$-modules over the group algebra $\mathcal{H}(G /\!/ K)[H]$

$$\begin{tikzcd} \mathbb{Z}[G/K] \arrow[r, two heads, "\phi"] \arrow[rr, bend left, two heads, "\phi_0"] & \mathbb{Z}[H^{\text{der}}\backslash G/K] \arrow[r, two heads] & \mathbb{Z}[H_0\backslash G/K] \end{tikzcd}$$

where $H^{\mathrm{der}} := \mathbf{H}^{\mathrm{der}}(F) = \Delta(\mathrm{SL}_n)(F)$ and $H_0 \subset H$ is the normal subgroup $\det^{-1}(\mathcal{O}_F^\times) \supset H^{\mathrm{der}}$.

Theorem 2.2.1. *For every integers $m \geq 1, k \geq 0$, we have*

(1) *Set $\tilde{H}_m := \mathrm{Stab}_H(v_m)$, then $\det(\tilde{H}_m) = 1 + \varpi^m \mathcal{O}_F$, thus the conductor of $\psi(v_m) \in H^{\mathrm{der}} \backslash G/K$ is m. We write as usual $H_m := \det(1+\varpi^m\mathcal{O}_F)^{-1} = \tilde{H}_m H^{der}$.*

(2) *Write $Succ_k(v_m)$ for the set of vertices appearing in the support of $\mathcal{U}_{\overline{\eta}(\varpi^k)}(v_m)$. For every $v \in Succ_k(v_m)$, we have $\phi(v) = H^{\mathrm{der}}(\widetilde{u}_v, 1)\overline{\eta}(\varpi^{m+k})v_0$ such that*

$$\widetilde{u}_v \in A := \begin{pmatrix} 1 & & & 1+\varpi^m\mathcal{O}_F \\ & \ddots & & \vdots \\ & & 1 & 1+\varpi^m\mathcal{O}_F \\ & & & 1 \end{pmatrix}$$

(3) *For every $v \in Succ_k(v_m)$, one has*

$$H_m \cdot \phi(v) = H_m \cdot \phi(v_{m+k}).$$

(4) *We have*

$$\phi(\mathcal{U}_{\overline{\eta}(\varpi^k)} v_m) \in \mathbb{Z}[H^{der} \backslash G/K]^{H_m}.$$

(5) *Set $\gamma = n(n-1)/2$. We have*

$$\phi(\mathcal{U}_{\overline{\eta}(\varpi^k)} v_m) = q^{\gamma k} \sum_{[h] \in H_m/H_{m+k}} h \cdot \phi(v_{m+k}).$$

Proof. (1) Lemma 4.4.3 gives in our particular case

$$\tilde{H}_m = \begin{pmatrix} 1+\varpi^m\mathcal{O}_F & \varpi^m\mathcal{O}_F & \varpi^{2m}\mathcal{O}_F & \cdots & \varpi^{(n-1)m}\mathcal{O}_F \\ \varpi^m\mathcal{O}_F & \ddots & \ddots & \ddots & \vdots \\ \varpi^{2m}\mathcal{O}_F & \ddots & \ddots & \ddots & \varpi^{2m}\mathcal{O}_F \\ \vdots & \ddots & \ddots & \ddots & \varpi^m\mathcal{O}_F \\ \varpi^{(n-1)m}\mathcal{O}_F & \cdots & \varpi^{2m}\mathcal{O}_F & \varpi^m\mathcal{O}_F & 1+\varpi^m\mathcal{O}_F \end{pmatrix}, \tag{2.1}$$

Accordingly, $\det(\tilde{H}_m) = 1 + \varpi^m\mathcal{O}_F$ by Proposition 4.4.4.

(2) Let $v \in \mathrm{Succ}_k(v_m)$ and write $v = (u_0, 1)\overline{\eta}(\varpi^m)(u_1, u_2)\overline{\eta}(\varpi^k)v_0$, for some $u_? \in I_?^+$ $(? = 1, 2)$. It follows

$$\begin{aligned} \phi(v) &= H^{\mathrm{der}}(u_0, 1)\overline{\eta}(\varpi^m)(u_1, u_2)\overline{\eta}(\varpi^k)v_0 \\ &= H^{\mathrm{der}}(u_0\eta(\varpi^m)u_1, \eta(\varpi^{-m})u_2)\overline{\eta}(\varpi^k)v_0 \\ &= H^{\mathrm{der}}(\eta(\varpi^{-m})u_2^{-1}\eta(\varpi^m)u_0\eta(\varpi^m)u_1, \eta(\varpi^{-m}))\overline{\eta}(\varpi^k)v_0 \\ &= H^{\mathrm{der}}(\iota(u_{2,m}^{-1})u_0u_{1,m}, 1)\overline{\eta}(\varpi^{m+k})v_0, \end{aligned}$$

where we have used the notation $u_{1,m} := \eta(\varpi^m)u_1\eta(\varpi^{-m}) \in I_1^+$ and $u_{2,m} := \eta(\varpi^{-m})u_2\eta(\varpi^m) \in I_2^+$. Observe that all the non diagonal entries of $u_{1,m}$ and $u_{2,m}$ are in $\varpi^m\mathcal{O}_F$. By writing explicitly the product $\iota(u_{2,m}^{-1})u_0u_{1,m}$ we see using (2.1) that it actually lies in

$$\iota(u_{2,m}^{-1})u_0u_{1,m} \in B = \begin{pmatrix} \boxed{\tilde{H}_m} & \begin{matrix} 1+\varpi^m\mathcal{O}_F \\ \vdots \\ 1+\varpi^m\mathcal{O}_F \end{matrix} \\ & 1 \end{pmatrix}$$

The subgroup $\tilde{H}_m \subset I_2$ is actually a subgroup of an element of the Moy–Prasad filtration [3], namely $H_{v_2,m} = (1+\varpi^m M_n(\mathcal{O}_F))$. By [MP96, Theorem 4.2] the group $H_{v_2,m}$ has an Iwahori decomposition with respect to B_2 and T_2

$$H_{v_2,m} = (H_{v_2,m} \cap U_2^-)(H_{v_2,m} \cap T_2)(H_{v_2,m} \cap U_2^+)$$

and this implies that

$$B \subset \iota\left(H_{v_2,m} \cap U_2^-\right) \cdot A \cdot \iota\left((H_{v_2,m} \cap T_2)(H_{v_2,m} \cap U_2^+)\right).$$

Accordingly, $\iota(u_{2,m}^{-1})u_0 u_{1,m} = \iota(\widetilde{u}_{2,m})\widetilde{u}_v t_v \widetilde{u}_{1,m}, 1)$, for some $\widetilde{u}_{2,m}, t_v, \widetilde{u}_{1,m}$ and $\widetilde{u}_v$ in $H_{v_2,m} \cap U_2^- \subset I_2^-$, $H_{v_2,m} \cap T_2$, $\iota(H_{v_2,m} \cap U_2^+) \subset I_1^-$ and A, respectively. We then conclude

$$\begin{aligned}\phi(v) &= H^{\mathrm{der}}(\iota(\widetilde{u}_{2,m})\widetilde{u}_v t_v \widetilde{u}_{1,m}, 1)\overline{\eta}(\varpi^{m+k})v_0 \\ &= H^{\mathrm{der}}(\widetilde{u}_v t_v \widetilde{u}_{1,m}, \widetilde{u}_{2,m}^{-1})\overline{\eta}(\varpi^{m+k})v_0 \\ &= H^{\mathrm{der}}(\widetilde{u}_v, 1)\overline{\eta}(\varpi^{m+k})v_0.\end{aligned}$$

(3) Write

$$\widetilde{u}_v = \begin{pmatrix} 1 & & & c_1 \\ & \ddots & & \vdots \\ & & 1 & c_n \\ & & & 1 \end{pmatrix}$$

Set $\underline{c} := \mathrm{diag}(c_1, \cdots, c_n)$, then $\underline{c} \in H_m$ and accordingly

$$\Delta(\underline{c}^{-1}) \cdot \phi(v) = H^{\mathrm{der}}(u_0, 1)\overline{\eta}(\varpi^{m+k})v_0 = \phi(v_{m+k})$$

and this shows that $H_m \cdot \phi(v) = H_m \cdot \phi(v_{m+k})$.

(4) For every $h \in H_m$ let $d_h \in T$ be the diagonal matrice with first entry $\det(h)^{-1}$ Then, since $\mathcal{U}_{\overline{\eta}(\varpi^k)}$ is B-equivariant we get

$$h \cdot \phi(\mathcal{U}_{\overline{\eta}(\varpi^k)}v_m) = \phi(\frac{h}{d_h} d_h \mathcal{U}_{\overline{\eta}(\varpi^k)}v_m) = \phi(\mathcal{U}_{\overline{\eta}(\varpi^k)}d_h v_m)$$

All we need to verify is that $d_h \in H_m$ and this is true by the description of H_m in (2.1). Therefore, we have $\phi(\mathcal{U}_{\overline{\eta}(\varpi^k)}v_m) \in \mathbb{Z}[H^{\mathrm{der}}\backslash G/K]^{H_m}$.

(5) We deduce from the previous point that

$$\phi(\mathcal{U}_{\overline{\eta}(\varpi^k)}v_m) = \sum_{y \in H^{\mathrm{der}}\backslash G/K} a_y y = a_{v_{m+k}} \sum_{h \in H_m/H_{m+k}} h \cdot v_{m+k}.$$

[3] For each parahoric subgroup $\mathcal{P}_a \subset \mathcal{G}$ Moy and Prasad [MP94, §2] have constructed an exhaustive filtration of open-compact normal subgroups $\mathcal{P}_{a,r} \subset \mathcal{P}_a$ for all $r \geq 0$.

This shows, in particular, that all the fibers of $\psi\colon \mathrm{Succ}_k(v_m) \to \phi(\mathrm{Succ}_k(v_m))$ have the same order and therefore,

$$\begin{aligned}\phi(\mathcal{U}_{\overline{\eta}(\varpi^k)}v_m) &= \sum_{v\in \mathrm{Succ}_k(v_m)} \phi(v)\\ &= \frac{|\mathrm{Succ}_k(v_m)|}{|H_m/H_{m+k}|}\sum_{h\in H_m/H_{m+k}} \phi(h\cdot v_{m+k})\\ &= \frac{|I^+/I^+\overline{\eta}(\varpi^k)K\overline{\eta}(\varpi^{-k})|}{|H_m/H_{m+k}|}\sum_{h\in H_m/H_{m+k}} \phi(h\cdot v_{m+k})\end{aligned}$$

- The size of the H_m-orbit of v_{m+k}: By the explicit formula of (2.1), we have
$$|H_m/H_{m+k}| = q^{k(n+2\sum_{i=1}^{n-1} i(n-i))}.$$
- The size of $\mathrm{Succ}_k(v_m)$: By Lemma 2.1.1 one has
$$\begin{aligned}|I^+/I^+\overline{\eta}(\varpi^k)K\overline{\eta}(\varpi^{-k})| &= |I_1^+/I_1^+\iota\circ\eta(\varpi^k)K_1\iota\circ\eta(\varpi^{-k})|\cdot|I_2^+/I_2^+\eta(\varpi^{-k})K_2\eta(\varpi^k)|\\ &= q^{k\left(\sum_{i=1}^{n} i(n-i+1)+\sum_{i=1}^{n-1} i(n-i)\right)}.\end{aligned}$$

Finally, this proves

$$\phi(\mathcal{U}_{\overline{\eta}(\varpi^k)}v_m) = q^{kn(n-1)/2}\sum_{h\in H_m/H_{m+k}} \phi(h\cdot v_{m+k}). \quad \square$$

Let $H_{\overline{\eta}}(X) = \sum_{k=0}^{N} C_k X^k \in \mathcal{H}_K[\mathbb{Z}][X]$ be the minimal unitary polynomial annihilating the operator $\mathcal{U}_{\overline{\eta}}$, where N denotes $|W(T_1\times T_2)\cdot\overline{\eta}| = n!(n+1)!$[4] since $\overline{\eta}$ is regular. We obtain as a corollary the following local vertical distribution:

Corollary 2.2.2 (Local vertical norm-compatible system)**.** *The family of vertices $\{v_m\}_{m\geq 1}$ in the fixed apartment $\mathcal{A}_{\mathrm{ext}}$, verifies the following vertical distribution relation:*

$$\sum_{k=0}^{N} q^{k\gamma}C_k \sum_{h\in H_m/H_{m+k}} h\cdot\phi(v_{m+k}) = 0.$$

Proof. By Theorem [**?**, Theorem 2.8.3],

$$H_{\overline{\eta}}(\mathcal{U}_{\overline{\eta}}) = \sum_{k=1}^{N} C_k\mathcal{U}_{\overline{\eta}}^k = 0 \text{ in } \mathrm{End}_{\mathbb{Z}[P_{\overline{\eta}}]}\mathbb{Z}[G/K].$$

[4]The Weyl group is the product $W(T_1\times T_2) = N_{\mathrm{GL}(n+1)(F)}T_1/T_1 \times N_{\mathrm{GL}(n)(F)}T_2/T_2 \simeq \mathfrak{S}_{n+1}\times\mathfrak{S}_n$ (a product of permutation groups).

In particular, for all $m \geq 1$ one has $\sum_{k=0}^{N} C_k \mathcal{U}_{\eta}^{k}(v_m) = 0$, hence

$$\begin{aligned} 0 = \phi(\sum_{k=0}^{N} C_k \mathcal{U}_{\eta}^{k}(v_m)) &= \sum_{k=1}^{N} C_k \phi(\mathcal{U}_{\eta}^{k}(v_m)) && C_k \in \mathcal{H}_K \\ &= \sum_{k=0}^{N} q^{k\gamma} C_k \sum_{h \in H_m/H_{m+k}} \phi(h \cdot v_{m+k}) \\ &= \sum_{k=0}^{N} q^{k\gamma} C_k \sum_{h \in H_m/H_{m+k}} h \cdot \phi(v_{m+k}) && \phi \text{ is } H\text{-equivariant.} \end{aligned}$$ □

For global consideration, we will write the above $g_m, m \geq 1$ as $g_{v,m}$ where v is the fixed place of F.

3. Global vertical distribution relations

3.1. **Global notations.** Set $\mathcal{M} = \mathcal{M}_{sp} \cdot \mathcal{M}_{in}$ for the set of products (not necessarily square-free) of prime ideals of F in $\mathcal{P}$.

Lemma 3.1.1. *Let $\mathfrak{f} = \prod_i \mathfrak{p}_i^{c_i} \in \mathcal{M}$ and $\mathfrak{p}$ a prime dividing $\mathfrak{f}$ and v the place of F corresponding to it. The extension $\mathcal{K}(\mathfrak{p}\mathfrak{f})/\mathcal{K}(\mathfrak{f})$ is of degree q_v^n.*

Proof. This is a special case of Corollary 3.9.6, which says

$$\mathrm{Gal}(\mathcal{K}(\mathfrak{p}^n\mathfrak{f})/\mathcal{K}(\mathfrak{f})) \simeq \frac{\mathfrak{O}_{\mathfrak{f}}^{\times}}{\mathfrak{O}_{\mathfrak{p}\mathfrak{f}}^{\times}} \simeq \frac{\mathcal{O}_{v,\mathrm{ord}_{F_v}(\mathfrak{f})}^{\times}}{\mathcal{O}_{v,\mathrm{ord}_{F_v}(\mathfrak{f})+n}^{\times}},$$

where v is the place of F corresponding to $\mathfrak{p}$. Thus, by Lemma 3.10.2

$$|\mathrm{Gal}(\mathcal{K}(\mathfrak{p}^n\mathfrak{f})/\mathcal{K}(\mathfrak{f}))| = |\left(\mathbb{F}^k(v)[\epsilon]\right)^{\times}/\mathbb{F}^k(v)^{\times}| = |\mathbb{G}_v(\epsilon)|^n = q_v^n. \quad \square$$

3.2. **Definition of the vertical norm-compatible system.** For any $\mathfrak{f} \in \mathcal{M}_{sp}^r$, set $\mathcal{P}_{\mathfrak{f}} \subset \mathcal{P}_{sc}$ for the places of F defined by the prime ideals dividing $\mathfrak{f}$. Define

$$\xi_{\mathfrak{f}} := \pi_{\mathrm{cyc}}\big([g_{0,S}] \otimes [g_{\mathfrak{f}}]^S\big) \in \mathbb{Q}[\mathcal{Z}_{\mathbf{G},K}(\mathbf{H})]$$

where, $[g_{0,S}] := \mathbf{H}^{\mathrm{der}}(F_S) g_{0,S} K_S$, $[g_{\mathfrak{f}}]^S := \big(\otimes_{v \in \mathcal{P}_{\mathfrak{f}}} [g_{v,\mathrm{ord}_{F_v}(\mathfrak{f})}]\big) \otimes \big(\otimes_{v \notin S \cup \mathcal{P}_{\mathfrak{f}}} [1]_v\big)$.

Proposition 3.2.1. *For each $\mathfrak{f} \in \mathcal{M}_{sp}$, the field of definition $E_{\mathfrak{f}}$ of $\xi_{\mathfrak{f}}$ is contained in $\mathcal{K}(\mathfrak{f})$ the $\mathcal{K}$-transfer field of conductor $\mathfrak{f}$.*

Proof. The proof of Proposition 4.6.1 is also valid for non square free product of ideals in $\mathcal{P}$. □

Remark 3.2.2. Proposition 3.2.1 and Lemma 3.9.3 imply that $E_{\mathfrak{f}}$ is also contained in $E(\mathfrak{c}_1\mathfrak{f})$. Therefore, by Corollary 3.9.4 any prime ideal of E above an ideal $\mathfrak{p} \in \mathcal{P}$ that is prime to $\mathfrak{f}$ is unramified in the extension $E_{\mathfrak{f}}/E$, i.e.

$$E_{\mathfrak{f}} \subset \mathcal{K}(\mathfrak{f}) \subset E(\infty)^{un,w_{\mathfrak{p}}}.$$

3.3. **Global vertical compatibility (Proof of Theorem 1.2.1).** For any non trivial $\mathfrak{f} \in \mathcal{M}$, let $v_\circ \in \mathcal{P}_\mathfrak{f}$ with prime ideal $\mathfrak{p}_{v_\circ}$. Let $H_{\overline{\eta}_{v_\circ}} = \sum_{k=0} C_{v_\circ,k} X^k \in \mathcal{H}_{K_{\mathfrak{p}_{v_\circ}}}[\mathbb{Z}][X]$ be the minimal unitary polynomial annihilating the operator $\mathcal{U}_{\overline{\eta}_{v_\circ}}$. Set $\mathrm{ord}_{v_\circ}(\mathfrak{f}) := \mathrm{ord}_{F_{v_\circ}}(\mathfrak{f})$.

$$\begin{aligned}
\sum_{k=0} q^{\gamma k} C_{v_\circ,k}\, \mathrm{Tr}_{\mathcal{K}(\mathfrak{p}_{v_\circ}^k \mathfrak{f})/\mathcal{K}(\mathfrak{f})}\, \xi_{\mathfrak{p}_{v_\circ}^k \mathfrak{f}} &\overset{\text{(Pro. 3.1.1)}}{=} \sum_{k=0} q^{\gamma k} C_{v_\circ,k}\, \mathrm{Tr}_{\mathcal{K}(\mathfrak{p}_{v_\circ}^k \mathfrak{f})_{w_\circ}/\mathcal{K}(\mathfrak{f})_{w_\circ}}\, \xi_{\mathfrak{p}_{v_\circ}^k \mathfrak{f}} \\
&= \sum_{k=0} q^{\gamma k} C_{v_\circ,k} \sum_{\lambda \in \mathcal{O}^\times_{v_\circ,\mathrm{ord}_{v_\circ}(\mathfrak{f})+k}/\mathcal{O}^\times_{v_\circ,\mathrm{ord}_{v_\circ}(\mathfrak{f})}} \xi_{\mathfrak{p}_{v_\circ}^k \mathfrak{f}}^{\mathrm{Art}_{w_\circ}(\lambda)} \\
&\overset{\text{(Pro. 3.4.1)}}{=} \sum_{\lambda \in \mathcal{O}^\times_{v_\circ,\mathrm{ord}_{v_\circ}(\mathfrak{f})+k}/\mathcal{O}^\times_{v_\circ,\mathrm{ord}_{v_\circ}(\mathfrak{f})}} \pi_{\mathrm{cyc}}\Big(\lambda \cdot \big([g_{0,s}] \otimes [g_{v_\circ,\mathrm{ord}_{v_\circ}(\mathfrak{f})}] \otimes [g_{\mathfrak{f}/\mathfrak{p}^{\mathrm{ord}_{v_\circ}(\mathfrak{f})}}]^{S\cup\{v_\circ\}}\big)\Big) \\
&= \sum_{k=0} q^{\gamma k} C_{v_\circ,k} \pi_{\mathrm{cyc}} \left(([g_{0,s}] \otimes \sum_{\lambda \in \mathcal{O}^\times_{v_\circ,\mathrm{ord}_{v_\circ}(\mathfrak{f})+k}/\mathcal{O}^\times_{v_\circ,\mathrm{ord}_{v_\circ}(\mathfrak{f})}} \lambda \cdot [g_{v_\circ,\mathrm{ord}_{v_\circ}(\mathfrak{f})}] \otimes [g_{\mathfrak{f}/\mathfrak{p}^{\mathrm{ord}_{v_\circ}(\mathfrak{f})}}]^{S\cup\{v_\circ\}}) \right. \\
&= \pi_{\mathrm{cyc}} \left(\sum_{k=0} q^{\gamma k} C_{v_\circ,k} ([g_{0,s}] \otimes \sum_{\lambda \in \mathcal{O}^\times_{v_\circ,\mathrm{ord}_{v_\circ}(\mathfrak{f})+k}/\mathcal{O}^\times_{v_\circ,\mathrm{ord}_{v_\circ}(\mathfrak{f})}} \lambda \cdot [g_{v_\circ,\mathrm{ord}_{v_\circ}(\mathfrak{f})}] \otimes [g_{\mathfrak{f}/\mathfrak{p}^{\mathrm{ord}_{v_\circ}(\mathfrak{f})}}]^{S\cup\{v_\circ\}}) \right. \\
&\overset{\text{(Cor. 2.2.2)}}{=} 0
\end{aligned}$$

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