THE CENTER OF HECKE ALGEBRAS OF TYPES

REDA BOUMASMOUD, RADHIKA GANAPATHY

Abstract. We describe the center of the Hecke algebra of a type attached to a Bernstein block under some hypothesis. When $\mathbf{G}$ is a connected reductive group over non-archimedean local field F that splits over a tamely ramified extension of F and the residue characteristic of F does not divide the order of the absolute Weyl group of $\mathbf{G}$, the works of Kim-Yu and Fintzen associate a type to each Bernstein block and our hypothesis is satisfied for such types. We use our results to give a description of the Bernstein center of the Hecke algebra $\mathcal{H}(\mathbf{G}(F), K)$ when K belongs to a nice family of compact open subgroups of $\mathbf{G}(F)$ (which includes all the Moy–Prasad filtrations of an Iwahori subgroup) via the theory of types.

Introduction

Let F be a non-archimedean local field. For a connected, reductive group $\mathbf{G}$ over F, we write G for its F-points.

Let $\mathfrak{R}(G)$ denote the category of smooth, complex representations of G. Let $\mathfrak{B}(G)$ for the set of all inertial equivalence classes in G (this definition is recalled in Section 1.1). The Bernstein decomposition yields

$$\mathfrak{R}(G) = \prod_{\mathfrak{s}\in\mathfrak{B}(G)} \mathfrak{R}^{\mathfrak{s}}(G).$$

We are interested in describing the center of $\mathfrak{R}^{\mathfrak{s}}(G)$, $\mathfrak{s} \in \mathfrak{B}(G)$. Let J be a compact open subgroup of G and let ρ be an irreducible representation of J such that (J,ρ) is an $\mathfrak{s}$-type (see Definition 1.1). Then the category $\mathfrak{R}^{\mathfrak{s}}(G)$ is equivalent to $\mathcal{H}(G,\rho)$ – mod. This leads us to the question of understanding the center of Hecke algebras of types.

First, suppose π is an irreducible supercuspidal representation of G of the form $\mathrm{ind}_{\widetilde{J}}^{G}\tilde{\rho}$, where $\widetilde{J}$ is an open, compact mod center subgroup of G and $\tilde{\rho}$ is an irreducible representation of $\widetilde{J}$. Let ${}^0G$ be the open normal subgroup of G as in (1.1) and let $J =^0 G \cap \widetilde{J}$ and let ρ be an irreducible summand of $\tilde{\rho}|_J$. Then (J,ρ) is an $\mathfrak{s} = [G,\pi]_G$-type. Assume that the intertwiners of ρ, denoted $\mathcal{I}_G(\rho)$, is contained in $\widetilde{J}$. These requirements are satisfied for supercuspidal representations arising out of Yu's construction (see [25]), which exhaust all supercuspidal representations of G by [12] under the hypothesis that $\mathbf{G}$ splits over a tamely ramified extension of F and the residue characteristic of F does not divide the order of the Weyl group of $\mathbf{G}$. Let π_0 be an irreducible summand of $\pi|_{{}^0G}$. We show in Theorem 3.10 that multiplicity with which π_0 occurs in $\pi|_{{}^0G}$ is equal to the multiplicity with which ρ occurs in $\tilde{\rho}|_J$. In fact, this theorem holds for slightly more general open normal subgroups ${}^\flat G$ of G. Next, we go on to describe the center of $\mathcal{H}(G,\rho)$ of the type

2000 *Mathematics Subject Classification.* 11F70, 22E50.

(J,ρ): we show that the center $\mathcal{Z}(\mathcal{H}(G,\rho)) \simeq \mathbb{C}[{}^{\dagger}J/J]$ where ${}^{\dagger}J = \bigcap_{\nu \in X_{\widetilde{J}}(\tilde{\rho})} \ker(\nu)$, with $X_{\widetilde{J}}(\tilde{\rho}) = \{\nu \in \mathrm{Hom}(\widetilde{J}/J, \mathbb{C}^{\times}) \mid \tilde{\rho} \otimes \nu \simeq \tilde{\rho}\}$ (See Lemma 2.2, Lemma 3.12 and Theorem 3.13). We deduce that the Hecke algebra $\mathcal{H}(G,\rho)$ is commutative if and only if $\pi|_{{}^0G}$ is multiplicity free (see Proposition 3.14). We then discuss some applications of this result to Yu's supercuspidal representations in Section 7.1.

Next, we describe the center of non-supercuspidal blocks. Let $\mathfrak{s} = [M,\sigma]_G$ and $\mathfrak{s}_M = [M,\sigma]_M$. We assume that σ is an irreducible supercuspidal representation of M of the form $\mathrm{ind}_{\widetilde{J}_M}^{M} \tilde{\rho}_M$, where $\widetilde{J}_M$ is an open, compact mod center subgroup of M and $\tilde{\rho}_M$ is an irreducible representation of $\widetilde{J}_M$. Let (J_M, ρ_M) be the $\mathfrak{s}_M$-type as before. Again we assume that $\mathcal{I}_G(\rho_M) \subset \widetilde{J}_M$. Let (J,ρ) be a G-cover of (J_M,ρ_M). Then (J,ρ) is an $\mathfrak{s}$-type. We show in Theorem 4.8 that that the center $\mathcal{Z}(\mathcal{H}(G,\rho)) \simeq \mathbb{C}[{}^{\dagger}J_M/J_M]^{W(\rho_M)}$ where $W(\rho_M)$ is described in Proposition 4.5.

Now, assume that $\mathbf{G}$ splits over a tamely ramified extension of F and the residue characteristic of F does not divide the order of the absolute Weyl group of $\mathbf{G}$. Then by [14, 12], every Bernstein block has a Kim-Yu type attached to it and our results in the preceeding paragraphs hold for such types. We use this to give a description of the Bernstein center of $\mathcal{H}(G,K)$ for certain nice compact open subgroups K of G. Let us describe what these compact open subgroups are.

Let $\mathcal{B}(G,F)$ denote the Bruhat–Tits building of $\mathbf{G}$ over F. Let $\mathbf{S}$ be a maximal F-split torus in $\mathbf{G}$ and let $\mathcal{A}(S,F)$ be the apartment of $\mathbf{S}$ over F. For a compact open subgroup K of G and let $\mathfrak{R}_K(G)$ be the full sub-category of $\mathfrak{R}(G)$ consisting of representations (π,V) that are generated by their K-fixed vectors. In [2, Section 3.7 - 3.9], the authors put criteria $\heartsuit_S$ (see Definition 5.1) on the compact open subgroup K and prove that the category $\mathfrak{R}_K(G)$ is closed under taking sub-quotients when K satisfies these criteria. They further show that if x is a special point in $\mathcal{A}(S,F)$, then $G_{x,r}$ satisfies $\heartsuit_S$ for each $r>0$. It was long expected that the category $\mathfrak{R}_K(G)$ is closed under taking sub-quotients whenever $K = G_{x,r}$ for all points $x \in \mathcal{B}(G,F)$ and all $r>0$. In [3], Bestvina–Savin put slightly different criteria $\spadesuit_S$ (see Definition 5.4) on the compact open subgroup K for which the category $\mathfrak{R}_K(G)$ is closed under taking sub-quotients. Further, they prove that $G_{x,r}$ satisfies $\spadesuit_S$ for each $x \in \mathcal{A}(S,F)$ and each $r>0$. If K satisfies $\heartsuit$ or $\spadesuit$, then there is a finite subset $\mathfrak{S}(K) \subset \mathfrak{B}(G)$ such that

$$\mathfrak{R}_K(G) = \prod_{\mathfrak{s} \in \mathfrak{S}(K)} \mathfrak{R}^{\mathfrak{s}}(G).$$

When K satisfies $\spadesuit_S$, it is easy to see that for a Levi subgroup $\mathbf{M}$ of $\mathbf{G}$ that contains $\mathbf{S}$, and a representation σ of M, if $(\mathrm{Ind}_P^G \sigma)^K \neq 0$, then $\sigma^{K'_M} \neq 0$ for a G-conjugate K' of K (that has an Iwahori factorization), where $\mathbf{P} = \mathbf{MN}$ is a parabolic subgroup of $\mathbf{G}$ with Levi $\mathbf{M}$ and $K_M = K \cap P / K \cap N$. On the other hand, when K satisfies $\heartsuit_S$ it follows that if $(\mathrm{Ind}_P^G \sigma)^K \neq 0$, then $\sigma^{K_M} \neq 0$. This property yields finer information about the set $\mathfrak{S}(K)$ (see Lemma 5.3). For this reason, it is helpful to know which $G_{x,r}, x \in \mathcal{A}(S,F), r>0$, also satisfy $\heartsuit_S$. We prove two results in this direction. First, we show that if $\mathscr{a}$ is an alcove in $\mathcal{A}(S,F)$ and $x \in \mathscr{a}$, then $G_{x,m}, m \in \mathbb{N}$, always satisfies $\heartsuit_S$ (See Proposition 5.8 for the precise statement). Next, we give two examples of Moy-Prasad filtration subgroups that don't satisfy $\heartsuit_S$. The first one is the Moy-Prasad filtration subgroup $G_{x,1} \subset \mathrm{GL}_3(F)$, where x is a non-special point in the boundary of an alcove of $\mathcal{A}(S,F)$. The second is the

Moy-Prasad filtration subgroup $G_{x_b,3/8} \subset \mathrm{GL}_4(F)$ where x_b is the barycenter of an alcove of $\mathcal{A}(S,F)$ (see Section 5.2).

In Section 7, we use the results in the preceeding sections and give a description of the Bernstein center of $\mathcal{H}(G,K)$ where K is a compact open subgroup of G that satisfies $\spadesuit_S$ or $\heartsuit_S$, using the theory of types.

Acknowledgements

This project has its origin in a note that Thomas Haines had shared with the first author which indicated that one could try to use the theory of types to describe the center of Hecke algebras at deeper level. We are grateful to him for this suggestion. We would like to thank Xuhua He, Dipendra Prasad, Marie-France Vigneras, and Jiu-Kang Yu for the helpful discussions. We owe a debt of gratitude to the referee for reading the paper carefully and for the numerous helpful suggestions and corrections, particularly on Theorem 4.8 and Proposition 5.8.

1. The Bernstein center

Let F be a non-archimedean local field and let $\mathbf{G}$ be a connected, reductive group over F. Let $G = \mathbf{G}(F)$. Let $\mathbf{Z}$ denote the center of $\mathbf{G}$.

In this paper, we will consider two induction functors and a restriction functor (see [24, Section I.5.1]):

- Ind denotes the usual induction functor.
- ind denotes the compact induction functor.
- Res denotes the restriction functor.

1.1. **The Bernstein decomposition.** Let $X_F(G)$ be the group of F-rational characters $\chi : G \to F^\times$ of G. For $\chi \in X_F(G)$ and $s \in \mathbb{C}$, we define a smooth one-dimensional representation $g \to |\chi(g)|_F^s$ of G. Let $X_{\mathrm{nr}}(G)$ be the group of unramified quasi-characters of G, generated by maps $G \to \mathbb{C}^\times$ of this form. We write

$$^0G = \bigcap_{\chi \in X_{\mathrm{nr}}(G)} \mathrm{Ker}(\chi). \tag{1.1}$$

The quotient $G/{}^0G$ is free abelian of finite rank and $X_{\mathrm{nr}}(G) = \mathrm{Hom}(G/{}^0G, \mathbb{C}^\times)$.

Let $\mathfrak{R}(G)$ the category of smooth, complex representations of G and $\mathfrak{Z}$ its center. Let $\mathrm{Irr}(G)$ the set of irreducible objects in $\mathfrak{R}(G)$.

We consider pairs (M,σ) where $\mathbf{M}$ is an F-Levi subgroup of G and σ is a supercuspidal representation of M. Two pairs (M_1,σ_1) and (M_2,σ_2) are inertially equivalent if there exist $g \in G$ and $\chi \in X_{\mathrm{nr}}(M_2)$ such that

$$\mathbf{M_2} = \mathbf{M_1}^g \text{ and } \sigma_1^g = \sigma_2 \otimes \chi.$$

Here $\mathbf{M_1}^g = g\mathbf{M_1}g^{-1}$ and $\sigma_1^g : x \to \sigma_1(g^{-1}xg)$, for $x \in M_1^g$. We write $[M,\sigma]_G$ for the inertial equivalence class of the pair (M,σ) and $\mathfrak{B}(G)$ for the set of all inertial equivalence classes in G.

Now, let π be an irreducible, smooth representation of G. There exists an F-parabolic subgroup $\mathbf{P}$ of G with Levi component M, and an irreducible, supercuspidal representation σ of M such that π occurs as an irreducible sub-quotient of the normalized parabolically induced representation $\mathrm{Ind}_P^G(\sigma)$. The representation

π determines a unique inertial equivalence class $[M,\sigma]_G$ which we denote as $\mathfrak{I}(\pi)$ and call it the inertial support of π (See [24, §II.2.20] for more properties).

For $\mathfrak{s} \in \mathfrak{B}(G)$ we define a full subcategory $\mathfrak{R}^{\mathfrak{s}}(G)$ of $\mathfrak{R}(G)$ as follows. Let $(\pi, V) \in \mathfrak{R}(G)$. Then $(\pi, V) \in \mathfrak{R}^{\mathfrak{s}}(G)$ if and only if every irreducible subquotient of π has inertial support $\mathfrak{s}$.

Let us recall some results on the Bernstein decomposition.

(1) We have $\mathfrak{R}(G) = \prod_{\mathfrak{s}\in\mathfrak{B}(G)} \mathfrak{R}^{\mathfrak{s}}(G)$. (see [21, Theorem 1.7.3.1].
(2) Let $\mathfrak{s} = [G,\pi]_G$ and let $\mathrm{Irr}^{\mathfrak{s}}(G)$ denote the set of isomorphism classes of irreducible objects in $\mathfrak{R}^{\mathfrak{s}}(G)$. Let $\mathfrak{Z}^{\mathfrak{s}}$ denote the center of $\mathfrak{R}^{\mathfrak{s}}(G)$. Then $\mathrm{Irr}^{\mathfrak{s}}(G)$ can be endowed with the structure of a complex affine variety whose ring of regular functions can be identified with the center $\mathfrak{Z}^{\mathfrak{s}}$ (see [21, Section 1.6.3]).
(3) Let $\mathfrak{t} = [M,\sigma]_M \in \mathfrak{B}(M)$ and let $\mathfrak{s} = [M,\sigma]_G \in \mathfrak{B}(G)$. The action of $N_G(M)$ on M by conjugation induces an action of $W(M)$ on $\mathfrak{B}(M)$. Let $W_{\mathfrak{t}}$ denote the stabilizer of $\mathfrak{t}$. Thus $W_{\mathfrak{t}} = N_{\mathfrak{t}}/M$ where
$$N_{\mathfrak{t}} = \{n \in N_G(M) \mid {}^n\sigma \simeq \sigma\nu, \text{ for some } \nu \in X_{\mathrm{nr}}(M)\}.$$
Let $\mathfrak{Z}^{\mathfrak{t}}$ denote the center of $\mathfrak{R}^{\mathfrak{t}}(M)$. The group $W^{\mathfrak{t}}$ acts on $\mathrm{Irr}^{\mathfrak{t}}(M)$ and hence acts on $\mathfrak{Z}^{\mathfrak{t}}$, the ring of regular functions of $\mathrm{Irr}^{\mathfrak{t}}(M)$. Let $\mathfrak{Z}^{\mathfrak{s}}$ denote the center of $\mathfrak{R}^{\mathfrak{s}}(G)$. Then (see [21, Theorem 1.9.1.1])
$$\mathfrak{Z}^{\mathfrak{s}} = (\mathfrak{Z}^{\mathfrak{t}})^{W_{\mathfrak{t}}}.$$

1.2. **Type associated to a Bernstein block.** Let $\Omega(G)$ the set of open compact subgroups of G and let $\Omega(G/Z)$ the set of open subgroups of G containing Z and compact mod Z. Let $\mathrm{Irr}(H)$ denote the set of irreducible representations of H for $H \in \Omega(G)$ or $\Omega(G/Z)$.

Definition 1.1. Let $J \in \Omega(G)$ and $\rho \in \mathrm{Irr}(J)$. For a subgroup H of G that contains J, let $\mathcal{H}(H,\rho) := \mathrm{End}_H(\mathrm{ind}_J^H(\rho))$.

The pair (J,ρ) is a type in G if it satisfies the following equivalent conditions (see [5, Section 4.2]):

(1) there exists a finite subset $\mathfrak{S}(J,\rho)$ of $\mathfrak{B}(G)$ such that for all $\pi \in \mathrm{Irr}(G)$, $\mathfrak{I}(\pi) \in \mathfrak{S}(J,\rho)$ if and only if $\mathrm{Hom}_J(\rho,\pi) \neq 0$;
(2) Let $\mathfrak{R}_\rho$ denotes the category of representations of G that are generated by their ρ-isotypic subspace. Then $\mathfrak{R}_\rho$ is closed under subquotients. Further
$$\mathfrak{R}_\rho = \mathfrak{R}^{\mathfrak{S}(J,\rho)}(G) := \prod_{\mathfrak{s}\in\mathfrak{S}(J,\rho)} \mathfrak{R}^{\mathfrak{s}}(G);$$
(3) the functor
$$\mathfrak{M}_\rho : \mathfrak{R}_\rho \to \mathcal{H}(G,\rho) - \mathrm{mod}, \quad \pi \longmapsto \mathrm{Hom}_J(\rho,\pi)$$
is an equivalence of categories, where $\mathcal{H}(G,\rho) - \mathrm{mod}$ denotes the category of modules over $\mathcal{H}(G,\rho)$.

In this situation, the pair (J,ρ) is called an $\mathfrak{S}(J,\rho)$-type.

Definition 1.2. We say that two types (J,ρ) and (J',ρ') are G-associate and write it as $(J,\rho) \approx_G (J',\rho')$ if $\mathfrak{S}(J,\rho) = \mathfrak{S}(J',\rho')$. The G-association class of a type (J,ρ) will be denoted $[J,\rho]_G$.

Remark 1.3. We will see later another different equivalence relation of (weakly cuspidal) types which will be denoted $\cong_G$ (see Definition 4.1).

2. Some Clifford theory

In this section, we collect some facts from Clifford theory that will be used later in the work. For the remainder of this paper, we will use the following notation. For a group H let $X(H) := \mathrm{Hom}(H, \mathbb{C}^\times)$. For a group H let $X(H) := \mathrm{Hom}(H, \mathbb{C}^\times)$. If $H \subset G$ and σ is a representation of H, then for any $g \in G$, σ^g denote the representation of $H^g = gHg^{-1}$ defined by $\sigma^g(ghg^{-1}) = \sigma(h)$, for all $h \in H$.

2.1. **$\flat$-world.** Let ${}^\flat G \trianglelefteq G$ of finite index in ${}^0G$ containing G^{der} and some open subgroup. So obviously, ${}^\flat G$ is open. In the latter part of the article, we will restrict to the case ${}^\flat G = {}^0G$. But for instance, one can take ${}^\flat G = \ker(\kappa_G)$ the kernel of the Kottwitz map.

2.2. Since ${}^\flat GZ$ is a normal subgroup of finite index in G and since for the representations considered below, Z acts via a character [24, II §2.8], we have the following as in usual Clifford theory.

Lemma 2.1. *Let $\tilde{H}$ be any open subgroup of G containing Z and set $H = \tilde{H} \cap {}^\flat G$. Let $\tilde{\sigma}$ be a semi-simple of finite length representation of $\tilde{H}$.*

(1) $\mathrm{Res}^{\tilde{H}}_H(\tilde{\sigma})$ is semi-simple of finite length so if H is compact then $\dim_{\mathbb{C}}(\tilde{\sigma}) < \infty$. Assume from now on that $\tilde{\sigma}$ is irreducible:

(2) Denote by $\mathcal{O}_H(\tilde{\sigma})$ the set of all $\sigma \in \mathrm{Irr}(H)$ which are isomorphic to a subrepresentation of $\mathrm{Res}^{\tilde{H}}_H(\tilde{\sigma})$. Write $\mathrm{Int}_{\widetilde{H}}(\sigma) = \{g \in \widetilde{H} \mid \sigma \simeq \sigma^g\}$ for the inertia group of σ in $\widetilde{H}$. We have $\mathcal{O}_H(\tilde{\sigma}) \simeq \widetilde{H}/\mathrm{Int}_{\widetilde{H}}(\sigma)$.

(3) Let $\sigma \in \mathrm{Irr}(H)$ contained in $\mathrm{Res}^{\tilde{H}}_H(\tilde{\sigma})$. There exists a positive integer $m^{\widetilde{H}}_H(\widetilde{\sigma})$ such that

$$\mathrm{Res}^{\widetilde{H}}_H(\widetilde{\sigma}) \simeq m_H(\widetilde{\sigma}) \cdot \Big(\bigoplus_{g \in \widetilde{H}/\mathrm{Int}_{\widetilde{H}}(\sigma)} \sigma^g\Big)$$

and $\{\sigma^g \colon g \in \widetilde{H}/\mathrm{Int}_{\widetilde{H}}(\sigma)\}$ are all nonisomorphic conjugates of σ. In particular, if $\dim_{\mathbb{C}}(\sigma) < \infty$ then

$$\dim_{\mathbb{C}}(\tilde{\sigma}) = |\widetilde{H}/\mathrm{Int}_{\widetilde{H}}(\sigma)| \cdot m_H(\tilde{\sigma}) \cdot \dim_{\mathbb{C}}(\sigma).$$

(4) Let $\widehat{\sigma}$ be the sum of all subrepresentations of $\mathrm{Res}^{\widetilde{H}}_H(\widetilde{\sigma})$ that are isomorphic to σ. Then $\widehat{\sigma}$ is an irreducible representation for $\mathrm{Int}_{\widetilde{H}}(\sigma)$ and

$$\mathrm{Res}^{\mathrm{Int}_{\widetilde{H}}(\sigma)}_H(\widehat{\sigma}) \simeq m_H(\widetilde{\sigma}) \cdot \sigma \quad \text{and} \quad \widetilde{\sigma} \simeq \mathrm{Ind}^{\widetilde{H}}_{\mathrm{Int}_{\widetilde{H}}(\sigma)}(\widehat{\sigma}).$$

(5) For any $\tilde{\sigma}' \in \mathrm{Irr}(\tilde{H})$, the following properties are equivalent:

i) $\tilde{\sigma}' \simeq \chi \otimes \tilde{\sigma}$ for some character $\chi \in X(\tilde{H}/H)$;

ii) $\mathrm{Res}^{\tilde{H}}_H(\tilde{\sigma}') \simeq \mathrm{Res}^{\tilde{H}}_H(\tilde{\sigma})$;

iii) $\mathcal{O}_H(\tilde{\sigma}) = \mathcal{O}_H(\tilde{\sigma}')$;

iv) $\mathcal{O}_H(\tilde{\sigma})$ and $\mathcal{O}_H(\tilde{\sigma}')$ has a direct irreducible factor that are isomorphic.

Proof. (1) It suffices to show the same statement if $\tilde{\sigma} \in \mathrm{Irr}(\tilde{H})$. The subgroup $\tilde{H}/HZ \hookrightarrow G/{}^{\flat}GZ$ is finite abelian group, so the restriction $\mathrm{Res}^{\tilde{H}}_{HZ}(\tilde{\sigma})$ is semi-simple of finite length [24, I.6.12]. As $\tilde{\sigma} \in \mathrm{Irr}(\tilde{H})$ has a central character, $\mathrm{Res}^{\tilde{H}}_{H}(\tilde{\sigma})$ is semi-simple of finite length. Now if H is compact then all objects of $\mathrm{Irr}(H)$ are finite dimensional and we deduce that $\mathrm{Res}^{\tilde{H}}_{H}(\tilde{\sigma})$ is finite dimensional.

(2) Clear since $\tilde{\sigma}$ is irreducible.

(3) Thanks to the previous point we know that

(4) Clear.

(5) The implications $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$ are clear. Let us show $(iv) \Rightarrow (i)$: The vector space $\mathrm{Hom}_H(\mathrm{Res}^{\tilde{H}}_{H}(\tilde{\sigma}), \mathrm{Res}^{\tilde{H}}_{H}(\tilde{\sigma}')) \neq 0$ can be naturally endowed with an action[1] of $\tilde{H}/H$. Now since this vector space is finite dimensional and that we are working over an algebraically closed field, $\tilde{H}/H$ must have an eigenvector $\chi \in X(\tilde{H}/H)$, i.e. $\mathrm{Hom}_{\tilde{H}}(\tilde{\sigma} \otimes \chi, \tilde{\sigma}') \neq 0$ or equivalently $\tilde{\sigma} \otimes \chi \simeq \tilde{\sigma}'$. □

2.2.1. Let ${}^{\dagger}H := \cap_{\chi \in X_{\tilde{H}}(\tilde{\sigma})} \ker(\chi)$ where $X_{\tilde{H}}(\tilde{\sigma}) := \{\chi \in X(\widetilde{H}/H) : \widetilde{\sigma} \otimes \chi \simeq \widetilde{\sigma}\}$. This is an open normal finite index subgroup of $\tilde{H}$. We choose an irreducible H-subspace W of $\tilde{\sigma}$ whose $\tilde{H}$-stabilizer ${}^{s}H$ is maximal. We let ${}^{s}\sigma$ denote the natural representation of ${}^{s}H$ on W. Replacing by a $\tilde{H}$-conjugate, we can assume that $\mathrm{Res}^{{}^{s}H}_{H}({}^{s}\sigma) = \sigma$.

Lemma 2.2. *Using the previous notation, we have the following properties.*

(1) $\mathrm{Res}^{\tilde{H}}_{{}^{s}H}\tilde{\sigma} \simeq \bigoplus_{h \in \tilde{H}/{}^{s}H} ({}^{s}\sigma)^{h}$.

(2) We have ${}^{\dagger}H \leqslant {}^{s}H \leqslant \mathrm{Int}_{\tilde{H}}(\sigma)$ *and* $[\mathrm{Int}_{\tilde{H}}(\sigma) : {}^{s}H] = [{}^{s}H : {}^{\dagger}H] = m_H(\tilde{\sigma})$.

(3) $\tilde{\sigma} \simeq \mathrm{ind}^{\tilde{H}}_{{}^{s}H}({}^{s}\sigma)$.

(4) The representation ${}^{\dagger}\sigma := \mathrm{Res}^{{}^{s}H}_{{}^{\dagger}H}({}^{s}\sigma)$ *is the unique irreducible representation which occurs in* $\mathrm{Res}^{\tilde{H}}_{{}^{\dagger}H}(\tilde{\sigma})$ *and satisfies* $\mathrm{Res}^{{}^{\dagger}H}_{H}({}^{\dagger}\sigma) = \sigma$.

(5) $\mathrm{ind}^{\tilde{H}}_{{}^{\dagger}H}({}^{\dagger}\sigma) \simeq m_H(\tilde{\sigma}) \cdot \tilde{\sigma}$.

(6) $X_{\tilde{H}}(\tilde{\sigma}) \simeq X(\tilde{H}/{}^{\dagger}H)$.

Proof. This is [8, Lemma 8.3] when $\widetilde{H} = G$ and $H = {}^{\flat}G = {}^{0}G$, whose detailed proof can be found in [21, Lemma 1.6.3.1]. The exact same proof works out in this more general set up. □

3. Some results on Hecke algebras of types

Definition 3.1. Define $\mathcal{J}_G$ to be the set of pairs (J, ρ) formed by $J \in \Omega(G)$ and an irreducible representation ρ of it, such that:

- There exists $\tilde{J} \in \Omega(G/Z)$ such that $J = \tilde{J} \cap {}^{\flat}G$ and $\tilde{\rho} \in \mathrm{Irr}(\tilde{J})$ containing ρ,
- $\mathcal{I}_G(\widetilde{\rho}) = \widetilde{J}$, or equivalently $\pi = \mathrm{ind}^{G}_{\tilde{J}}(\tilde{\rho})$ is irreducible (hence supercuspidal).

For any $(J, \rho) \in \mathcal{J}_G$, set ${}^{0}J = \tilde{J} \cap {}^{0}G$ (the unique maximal compact subgroup of $\tilde{J}$) and fix a representation ${}^{0}\rho \in \mathrm{Irr}({}^{0}J)$ that is contained in $\tilde{\rho}$ and contains $\rho \in \mathrm{Irr}(J)$. We will also consider the pair $({}^{\natural}J := \cap_{\chi \in X_{{}^{0}J}({}^{0}\rho)} \ker(\chi), {}^{\natural}\rho)$, this[2] is the

[1]Consider the action $g(\psi) = \tilde{\sigma}'(g)\psi\tilde{\sigma}(g)$, for all $g \in \tilde{H}$ and $\psi \in \mathrm{Hom}_H(\mathrm{Res}^{\tilde{H}}_{H}(\tilde{\sigma}), \mathrm{Res}^{\tilde{H}}_{H}(\tilde{\sigma}'))$.

[2]The superscript $\natural$ is there to differentiate this group from ${}^{\dagger}J := \cap_{\chi \in X_{\tilde{J}}(\tilde{\rho})} \ker(\chi)$ which is attached to $(\tilde{H}, \tilde{\sigma}) = (\tilde{J}, \tilde{\rho})$ in Lemma 2.2.

pair $({}^{\dagger}H, {}^{\dagger}\rho)$ attached to $(\tilde{H}, \tilde{\sigma}) = ({}^{0}J, {}^{0}\rho)$ in Lemma 2.2. For any $\nu \in X({}^{\natural}J/J)$, fix an extension $\bar{\nu} \in X(G/{}^{\flat}G)$ and let ${}^{0}\nu := \mathrm{Res}^{G}_{{}^{0}J}(\bar{\nu}) \in X({}^{0}J/J)$.

Then, we can adapt the case ${}^{\flat}G = {}^{0}G$ treated in [5, Proposition 5.4] in the following way:

Theorem 3.2. *Let $(J,\rho) \in \mathcal{J}_G$ and $(\widetilde{J}, \tilde{\rho})$ as in Definition 3.1.*
(1) For $\nu \in X({}^{\natural}J/J)$ the pair $({}^{0}J, {}^{0}\rho \otimes {}^{0}\nu)$ is a $[G, \pi \otimes \bar{\nu}]_G$-type.
(2) For any $\nu \in X({}^{\natural}J/J)$ we have

$$({}^{0}J, {}^{0}\rho \otimes {}^{0}\nu) \approx_G ({}^{\natural}J, {}^{\natural}\rho \otimes \nu).$$

(3) The pair (J,ρ) is a type in G such that

$$\mathfrak{S}(J,\rho) = \bigsqcup_{\nu \in X({}^{\natural}J/J)} \mathfrak{S}({}^{0}J, {}^{0}\rho \otimes {}^{0}\nu) = \{[G, \pi \otimes \bar{\nu}]_G : \nu \in X({}^{\natural}J/J)\}.$$

Proof. (1) Follows from [5, Proposition 5.4].

(2) By Lemma 2.2 we have $\mathrm{ind}^{{}^{0}J}_{{}^{\natural}J}({}^{\natural}\rho \otimes \nu) = m_J({}^{0}\rho) \cdot {}^{0}\rho \otimes {}^{0}\nu$, the two pairs $({}^{0}J, {}^{0}\rho \otimes {}^{0}\nu)$ and $({}^{\natural}J, {}^{\natural}\rho \otimes \nu)$ are associate in the sense of [5, (ii) §4.4]. Therefore, by [5, (i) Remark §4.5], since the first pair is a $[G, \pi \otimes \bar{\nu}]_G$-type so is the second pair.

(3) Using Frobenius reciprocity and Lemma 2.1 one shows that for any $\sigma \in \mathrm{Irr}(G)$:

$$\begin{aligned} \mathrm{Hom}_J(\rho, \mathrm{Res}^G_J(\sigma)) &\simeq \mathrm{Hom}_{{}^{\natural}J}(\mathrm{ind}^{{}^{\natural}J}_J(\rho), \mathrm{Res}^G_{{}^{\natural}J}(\sigma)) \\ &\simeq \mathrm{Hom}_{{}^{\natural}J}({}^{\natural}\rho \otimes \mathbb{C}[{}^{\natural}J/J], \mathrm{Res}^G_{{}^{\natural}J}(\sigma)) \\ &\simeq \bigoplus_{\nu \in X({}^{\natural}J/J)} \mathrm{Hom}_{{}^{\natural}J}({}^{\natural}\rho \otimes \nu, \mathrm{Res}^G_{{}^{\natural}J}(\sigma)) \end{aligned}$$

This shows that $\mathfrak{R}_\rho = \cup_{\nu \in X({}^{\natural}J/J)} \mathfrak{R}_{{}^{\natural}\rho \otimes \nu}$. Using (1) and noting that

$$[G, \pi \otimes \bar{\nu}]_G \neq [G, \pi \otimes \bar{\nu}']_G \Leftrightarrow \mathrm{Res}^G_{{}^{0}G}(\bar{\nu}^{-1}\bar{\nu}') \text{ is trivial } \Leftrightarrow \nu = \nu'.$$

We deduce then $\mathfrak{R}_\rho = \bigsqcup_{\nu \in X({}^{\natural}J/J)} \mathfrak{R}_{{}^{\natural}\rho \otimes \nu}(G) = \bigsqcup_{\nu \in X({}^{\natural}J/J)} \mathfrak{R}^{[G,\pi \otimes \bar{\nu}]_G}(G)$. □

Corollary 3.3. *Let $(J,\rho) \in \mathcal{J}_G$. Any irreducible sub-quotient of $\mathrm{ind}^G_J(\rho)$ is isomorphic to $\pi \otimes \chi$ for some $\chi \in X(G/{}^{\flat}G)$.*

Proof. We have already seen in Theorem 3.2 that (J,ρ) is a type, hence for any irreducible sub-quotient σ of $\mathrm{ind}^G_J(\rho)$ we have $\mathfrak{I}(\sigma) \in \mathfrak{S}(J,\rho)$ since $\mathrm{Hom}_G(\mathrm{ind}^G_J(\rho), \sigma) \neq 0$. Accordingly, we have $\mathfrak{I}(\sigma) = [G, \pi \otimes \bar{\nu}]_G$ for some $\nu \in X({}^{\natural}J/J)$, and so there exists $g \in G$ such that $\sigma = \pi^g \otimes \bar{\nu}\chi \simeq \pi \otimes \bar{\nu}\chi$ for some $\chi \in X(G/{}^{0}G)$. □

Remark 3.4. Let $(J,\rho) \in \mathcal{J}_G$. By Schur's lemma we have an equality $\mathcal{I}_G(\rho) \cap \widetilde{J} = \mathrm{Int}_{\widetilde{J}}(\rho)$. Hence, since $\tilde{\rho} \simeq \mathrm{Ind}^{\widetilde{J}}_{\mathrm{Int}_{\widetilde{J}}(\rho)}(\widehat{\rho})$ where $\widehat{\rho}$ is the irreducible representation defined in (4) Lemma 2.1, we may replace the pair $(\widetilde{J}, \tilde{\rho})$ by $(\mathrm{Int}_{\widetilde{J}}(\rho), \widehat{\rho})$. Accordingly, for any pair $(J,\rho) \in \mathcal{J}_G$ we may assume that $\widetilde{J} \subset \mathcal{I}_G(\rho)$.

Lemma 3.5. *Let $\tilde{J} \in \Omega(G/Z)$, $J = \tilde{J} \cap {}^{\flat}G$ and $\tilde{\rho} \in \mathrm{Irr}(\tilde{J})$. For any $\rho \in \mathcal{O}_J(\tilde{\rho})$, we have $\mathcal{I}_G(\widetilde{\rho}) \subset \widetilde{J}\mathcal{I}_G(\rho)\widetilde{J}$. In particular,*

$$\mathcal{I}_G(\rho) \subset \tilde{J} \Rightarrow \mathcal{I}_G(\tilde{\rho}) = \tilde{J} \textit{ i.e. } (J,\rho) \in \mathcal{J}_G.$$

Proof. If $g \in G$ satisfies $\mathrm{Hom}_{\tilde{J}\cap\tilde{J}^g}(\tilde{\rho},\tilde{\rho}^g) \neq 0$, then $\mathrm{Hom}_{J\cap J^g}(\mathrm{Res}^{\tilde{J}}_J(\tilde{\rho}),\mathrm{Res}^{\tilde{J}}_J(\tilde{\rho})^g) \neq 0$. Using Mackey and Clifford theories we deduce that there exists $h, h' \in \tilde{J}$ such that $h'gh^{-1} \in \mathcal{I}_G(\rho)$, which shows the desired equality. □

Definition 3.6. A pair $(J,\rho) \in \mathcal{J}_G$ (and its class $[J,\rho]_G$) will be called *weakly cuspidal* if (i) $\mathcal{I}_G(\rho) \subset \widetilde{J}$, and *cuspidal* if in addition (ii) $m_J(\tilde{\rho}) = 1$. Write $\mathcal{J}_G^{wc}$ for the set of weakly cuspidal types.

Remark 3.7. In view of Remark 3.4 we may assume that if (J,ρ) is weakly cuspidal, then $\widetilde{J} = \mathcal{I}_G(\rho)$ and $\mathrm{Res}^{\widetilde{J}}_J(\tilde{\rho}) = m_J(\tilde{\rho})\cdot\rho$.

For cuspidal types, it is shown in [5, Proposition 5.6] that the Hecke algebra $\mathcal{H}(G,\rho)$ is commutative. As noted in [5, §5.5], (i) and (ii) are satisfied when $G = {}^0G$. The existence of a type that satisfies (i) and (ii) is known when $G = \mathrm{GL}_n$ or its inner forms and when G is a classical group provided the residue characteristic of F is odd (see [6, 16]).

3.1. Isomorphism of Hecke algebras.

Lemma 3.8. *Let $J \in \Omega(G)$ and $\rho \in \mathrm{Irr}(J)$. We have an isomorphism of algebras $\mathcal{H}(G,\rho) = \mathcal{H}(\mathcal{I}_G(\rho),\rho)$.*

Proof. The proof given here is essentially the same as the one in [5, Proposition 5.6]. We write the details for completeness.

Let $f \in \mathcal{H}(\mathcal{I}_G(\rho),\rho)$; we view elements on the Hecke algebra as functions as in [5, Section 2.1]. Define $\tilde{f}$ on $\mathcal{H}(G,\rho)$ by setting $\tilde{f}(x) = 0$ if $x \notin \mathcal{I}_G(\rho)$. The map $\Phi : \mathcal{H}(\mathcal{I}_G(\rho),\rho) \to \mathcal{H}(G,\rho)$, $f \to \tilde{f}$ is an algebra embedding. It remains to see that Φ is surjective. To prove this, it suffices to show that for $h \in \mathcal{H}(G,\rho)$, the support of h is contained in $\mathcal{I}_G(\rho)$. However, $g \in G$ lies in the support of a function in $\mathcal{H}(G,\rho)$ if and only if g intertwines ρ (see [5, Section 2.1]). This finishes the proof of the lemma. □

Lemma 3.9. *Let $(J,\rho) \in \mathcal{J}_G^{wc}$. The Hecke algebra $\mathcal{H}(G,\rho)$ is a free $\mathbb{C}[{}^\dagger J/J]$-module of rank $m_J(\tilde{\rho})^2$. In particular, if $m_J(\widetilde{\rho}) = 1$ then $\mathcal{H}(G,\rho) \simeq \mathbb{C}[{}^\dagger J/J]$ is commutative.*

Proof. We have the following isomorphism of $\mathbb{C}$-modules

$$\begin{aligned}
\mathcal{H}(G,\rho) &= \mathrm{End}_{\mathcal{I}_G(\rho)}(\mathrm{ind}_J^{\mathcal{I}_G(\rho)}(\rho)) &(3.1)\\
&\simeq \mathrm{Hom}_{{}^\dagger J}\left(\mathrm{Res}^{\mathcal{I}_G(\rho)}_{{}^\dagger J}\circ\mathrm{ind}_J^{\mathcal{I}_G(\rho)}(\rho), {}^\dagger\rho\otimes\mathbb{C}[{}^\dagger J/J]\right) &(3.2)\\
&\simeq \bigoplus_{j\in\mathcal{I}_G(\rho)/{}^\dagger J}\mathrm{Hom}_{{}^\dagger J}\left(\mathrm{ind}_J^{{}^\dagger J}(\rho^j), {}^\dagger\rho\otimes\mathbb{C}[{}^\dagger J/J]\right) &(3.3)\\
&\simeq m_J(\tilde{\rho})^2\,\mathrm{Hom}_{{}^\dagger J}({}^\dagger\rho\otimes\mathbb{C}[{}^\dagger J/J], {}^\dagger\rho\otimes\mathbb{C}[{}^\dagger J/J]) &(3.4)
\end{aligned}$$

The first isomorphism follows from Lemma 3.8, the second and fourth from Lemma 2.2 and the third from Mackey formula.

The map $\mu\colon \mathbb{C}[{}^\dagger J/J] \hookrightarrow \mathrm{End}_{{}^\dagger J}({}^\dagger\rho\otimes_{\mathbb{C}}\mathbb{C}[{}^\dagger J/J])$, that sends an element $\bar{w}$ to the endomorphism

$$v\otimes\bar{j} \longmapsto v\otimes\bar{j}\bar{w}^{-1}, \quad \forall v\in{}^\dagger\rho, \forall\bar{j}\in{}^\dagger J/J,$$

yields an embedding of $\mathbb{C}$-algebras

$$\mathbb{C}[{}^\dagger J/J] \hookrightarrow \mathrm{End}_{{}^\dagger J}({}^\dagger\rho\otimes_{\mathbb{C}}\mathbb{C}[{}^\dagger J/J]) \hookrightarrow \mathrm{End}_J(\rho\otimes_{\mathbb{C}}\mathbb{C}[{}^\dagger J/J]).$$

But since ρ is irreducible and J acts trivially on $\mathbb{C}[{}^\dagger J/J]$ we clearly have

$$\mathrm{End}_J(\rho\otimes_{\mathbb{C}}\mathbb{C}[{}^\dagger J/J]) \simeq \mathbb{C}[{}^\dagger J/J].$$

This shows that $\mu: \mathbb{C}[{}^\dagger J/J] \xrightarrow{\sim} \mathrm{End}_{{}^\dagger J}({}^\dagger\rho\otimes_{\mathbb{C}}\mathbb{C}[{}^\dagger J/J])$ is an isomorphism. The algebra $\mathbb{C}[{}^\dagger J/J]$ acts on $\mathrm{End}_{\mathcal{I}_G(\rho)}(\mathrm{ind}_J^{\mathcal{I}_G(\rho)}(\rho))$ as follows:

$$w\cdot\phi = \phi\circ \mathrm{ind}_{{}^\dagger J}^{\mathcal{I}_G(\rho)}(\mu(w)), \quad \forall w\in\mathbb{C}[{}^\dagger J/J],\ \forall\phi\in \mathrm{End}_{\mathcal{I}_G(\rho)}(\mathrm{ind}_J^{\mathcal{I}_G(\rho)}(\rho)).$$

Moreover, the isomorphisms in the equations below (3.1) are all $\mathbb{C}[{}^\dagger J/J]$-equivariant. This proves the lemma. □

3.2. **Multiplicites for types.** Let $(J,\rho)\in\mathcal{J}_G^{wc}$. In this subsection, we prove that $m_{{}^\flat G}(\pi) = m_J(\tilde\rho)$. We will deduce several consequences of this result in the subsequent subsections.

3.2.1. We recall [24, §8.3]: For any $K\in\Omega(G)$ and any $\sigma\in\mathrm{Irr}(K)$, the following statements are equivalent

(i) $\mathrm{ind}_K^G(\sigma)$ is irreducible,
(ii) $\mathrm{End}_G(\mathrm{ind}_K^G(\sigma)) = \mathbb{C}$,
(iii) $\mathcal{I}_G(\sigma) = K$,
(iv) σ is not contained in $\mathrm{ind}_{K\cap K^g}^K \mathrm{Res}_{K\cap K^g}^{K^g}(\sigma^g)$ for any $g\notin K$.

Consequently ${}^\flat\tilde\pi := \mathrm{ind}_{\tilde J}^{{}^\flat G\tilde J}(\tilde\rho)\in\mathrm{Irr}({}^\flat G\tilde J)$ and ${}^\flat\pi := \mathrm{ind}_J^{{}^\flat G}(\rho)\in\mathrm{Irr}({}^\flat G)$. For any $g\in G$, we have

$$\begin{aligned}\mathrm{Hom}_{{}^\flat G}({}^\flat\pi, {}^\flat\pi^g) &\simeq \mathrm{Hom}_{{}^\flat G}(\mathrm{ind}_J^{{}^\flat G}(\rho), \mathrm{ind}_{J^g}^{{}^\flat G}(\rho^g))\\ &\simeq \bigoplus_{h\in J^g\backslash {}^\flat G/J}\mathrm{Hom}_{J\cap J^{hg}}(\rho,\rho^{hg}).\end{aligned}$$

So since ${}^\flat\pi$ is irreducible the left Hom space is non zero if and only if $g\in{}^\flat G\mathcal{I}_G(\rho)$. Accordingly,

$$\mathrm{Int}_G({}^\flat\pi^g) = \mathcal{I}_G({}^\flat\pi^g) = {}^\flat G\mathcal{I}_G(\rho)^g, \quad \forall g\in G. \tag{3.5}$$

3.2.2.

Theorem 3.10. *We have*

$$m_{{}^\flat G}(\pi) = m_{{}^\flat G}({}^\flat\tilde\pi) = m_J(\tilde\rho).$$

Proof. Using Mackey theory we can easily see that ${}^\flat\tilde\pi$ contains ${}^\flat\pi$ and that π contains ${}^\flat\tilde\pi$, so given (3.5) and using Lemma 2.1 we have

$$\mathrm{Res}_{{}^\flat G}^{{}^\flat G\tilde J}({}^\flat\tilde\pi) = m_{{}^\flat G}({}^\flat\tilde\pi)\bigoplus_{h\in\tilde J/\mathcal{I}_G(\rho)}{}^\flat\pi^h \text{ and } \mathrm{Res}_{{}^\flat G\tilde J}^G(\pi) = m_{{}^\flat G\tilde J}(\pi)\bigoplus_{h\in G/{}^\flat G\tilde J}{}^\flat\tilde\pi^h.$$

Similarly

$$\mathrm{Res}_{{}^\flat G}^G(\pi) = m_{{}^\flat G}(\pi)\bigoplus_{h\in G/{}^\flat G\mathcal{I}_G(\rho)}{}^\flat\pi^h.$$

We are going to compute the dimension of $\Phi := \mathrm{Hom}_G(\mathrm{ind}_J^G(\rho), \mathrm{ind}_{\tilde J}^G(\widetilde{\rho}))$ in various ways, mainly by playing with Mackey Theory and Frobenius reciprocities.

- First, $\Phi\simeq\mathrm{Hom}_{{}^\flat G}({}^\flat\pi, \mathrm{Res}_{{}^\flat G}^G(\pi))$. So by the irreducibility of ${}^\flat\pi$, and Clifford theory

$$\dim_{\mathbb{C}}(\Phi) = \bigoplus_{G/\mathrm{Int}_G({}^\flat\tilde\pi)} m_{{}^\flat G}(\pi)\dim_{\mathbb{C}}(\mathrm{Hom}_{{}^\flat G}({}^\flat\pi, ({}^\flat\pi)^h) = m_{{}^\flat G}(\pi). \tag{3.6}$$

- Recall that $G/{}^\flat G\tilde{J}$ is finite, so $\mathrm{ind}_{{}^\flat G\tilde{J}}^{G}({}^\flat\tilde{\pi}) = \mathrm{Ind}_{{}^\flat G\tilde{J}}^{G}({}^\flat\tilde{\pi})$ is admissible. Using Frobenius reciprocity and Mackey theory

$$\begin{aligned}\Phi &\simeq \mathrm{Hom}_{{}^\flat G\tilde{J}}(\mathrm{Res}_{{}^\flat G\tilde{J}}^{G}(\mathrm{ind}_{J}^{G}(\rho)), {}^\flat\tilde{\pi}) \\ &\simeq \bigoplus_{g\in J\backslash G/{}^\flat G\tilde{J}} \mathrm{Hom}_{{}^\flat G\tilde{J}}(\mathrm{ind}_{{}^\flat G\tilde{J}\cap J^g}^{{}^\flat G\tilde{J}} \mathrm{Res}_{J^g}^{{}^\flat G\tilde{J}\cap J^g}(\rho^g), {}^\flat\tilde{\pi}) \\ &\simeq \bigoplus_{g\in G/{}^\flat G\tilde{J}} \mathrm{Hom}_{{}^\flat G\tilde{J}}(\mathrm{ind}_{J^g}^{{}^\flat G\tilde{J}}(\rho^g), {}^\flat\tilde{\pi})\end{aligned}$$

Observe that $\mathrm{ind}_{J^g}^{{}^\flat G\tilde{J}}(\rho^g) = \mathrm{ind}_{{}^\flat G}^{{}^\flat G\tilde{J}}\mathrm{ind}_{J^g}^{{}^\flat G}(\rho^g) = \mathrm{ind}_{{}^\flat G}^{{}^\flat G\tilde{J}}({}^\flat\pi^g)$. So

$$\begin{aligned}\Phi &\simeq \bigoplus_{g\in G/{}^\flat G\tilde{J}} \mathrm{Hom}_{{}^\flat G\tilde{J}}(\mathrm{ind}_{{}^\flat G}^{{}^\flat G\tilde{J}}({}^\flat\pi^g), {}^\flat\tilde{\pi}) \\ &\simeq \bigoplus_{g\in G/{}^\flat G\tilde{J}} \mathrm{Hom}_{{}^\flat G}({}^\flat\pi^g, \mathrm{Res}_{{}^\flat G}^{{}^\flat G\tilde{J}}({}^\flat\tilde{\pi})) \\ &\simeq m_{{}^\flat G}({}^\flat\tilde{\pi}) \bigoplus_{j\in{}^\flat G\tilde{J}/\mathrm{Int}_{{}^\flat G\tilde{J}}({}^\flat\pi)} \bigoplus_{g\in G/{}^\flat G\tilde{J}} \mathrm{Hom}_{{}^\flat G}({}^\flat\pi^g, {}^\flat\pi^j)\end{aligned}$$

For any $j \in \tilde{J}/\mathcal{I}_G(\rho)$, The last Hom space is nonzero only if $g^{-1}j \in \mathcal{I}_G({}^\flat\pi) = {}^\flat G\mathcal{I}_G(\rho)$, hence $g \in {}^\flat G\tilde{J} \subset \mathrm{Int}_G({}^\flat\tilde{\pi}^g)$. So,

$$\Phi \simeq m_{{}^\flat G}({}^\flat\tilde{\pi}) \bigoplus_{j\in\tilde{J}/\mathcal{I}_G(\rho)} \mathrm{Hom}_{{}^\flat G}({}^\flat\pi, {}^\flat\pi^j) = m_{{}^\flat G}({}^\flat\tilde{\pi})\,\mathrm{End}_{{}^\flat G}({}^\flat\pi)$$

Therefore,

$$\dim_{\mathbb{C}}(\Phi) = m_{{}^\flat G}({}^\flat\tilde{\pi}). \tag{3.7}$$

- Finally,

$$\begin{aligned}\Phi &\simeq \bigoplus_{g\in G/{}^\flat G\tilde{J}} \mathrm{Hom}_{{}^\flat G}({}^\flat\pi^g, \mathrm{Res}_{{}^\flat G}^{{}^\flat G\tilde{J}}({}^\flat\tilde{\pi})) \\ &\simeq \bigoplus_{g\in G/{}^\flat G\tilde{J}} \bigoplus_{g\in\tilde{J}\backslash{}^\flat G\tilde{J}/{}^\flat G} \mathrm{Hom}_{{}^\flat G}({}^\flat\pi^g, \mathrm{ind}_{{}^\flat G\cap\tilde{J}^h}^{{}^\flat G} \mathrm{Res}_{{}^\flat G\cap\tilde{J}^h}^{\tilde{J}^h}(\tilde{\rho}^h)) \\ &\simeq \bigoplus_{g\in G/{}^\flat G\tilde{J}} \mathrm{Hom}_{{}^\flat G}({}^\flat\pi^g, \mathrm{ind}_{J}^{{}^\flat G} \mathrm{Res}_{J}^{\tilde{J}}(\tilde{\rho})) \\ &\simeq m_J(\tilde{\rho}) \bigoplus_{g\in G/{}^\flat G\tilde{J}} \bigoplus_{j\in\tilde{J}/\mathcal{I}_G(\rho)} \mathrm{Hom}_{{}^\flat G}({}^\flat\pi^g, {}^\flat\pi^j)\end{aligned}$$

Accordingly $\Phi \simeq m_J(\tilde{\rho})\,\mathrm{End}_{{}^\flat G}({}^\flat\pi)$. In conclusion $\dim_{\mathbb{C}}(\Phi) = m_J(\tilde{\rho})$.

□

Corollary 3.11. *We have*

$$\mathrm{Res}_{{}^\flat G}(\pi) = m_J(\tilde{\rho}) \bigoplus_{h\in G/{}^\flat G\mathcal{I}_G(\rho)} {}^\flat\pi^h.$$

Lemma 3.12. *Let ${}^\dagger J$ be as in Lemma 2.2 where $\widetilde{H} = \widetilde{J}$ and $\tilde{\sigma} = \tilde{\rho}$. Then*

$$\bigcap_{\chi\in X_G(\pi)} \ker(\chi) = {}^\dagger J\,{}^\flat G$$

In particular, ${}^\dagger J = \mathcal{I}_G(\rho) \cap \bigcap_{\chi\in X_G(\pi)} \ker(\chi)$ *and* ${}^\dagger({}^\flat\pi) = \mathrm{ind}_{{}^\dagger J}^{{}^\dagger J\,{}^\flat G}({}^\dagger\rho)$.

Proof. Consider the inclusion $\psi: \mathcal{I}_G(\rho)/J \hookrightarrow G/{}^\flat G$. The subgroup $\psi(\mathcal{I}_G(\rho)/J)$ is a finite index subgroup of $G/{}^\flat G$. Hence, any character of $\mathcal{I}_G(\rho)/J$ can be extended to a character of $G/{}^\flat G$ in $[G/{}^\flat G : \psi(\mathcal{I}_G(\rho)/J)]$ ways. Write $\psi(X_{\tilde{J}}(\tilde{\rho}))$ for those that are extensions of element in $X_{\tilde{J}}(\tilde{\rho})$. Note that for any $\nu \in X_{\tilde{J}}(\tilde{\rho})$ if $\tilde{\rho} \simeq \tilde{\rho} \otimes \nu$ then $\pi \simeq \pi \otimes \bar{\nu}$ where $\bar{\nu}$ is any extension of ν to G. This shows that $\psi(X_{\tilde{J}}(\tilde{\rho})) \subset X_G(\pi)$ and so

$$\bigcap_{\chi \in X_G(\pi)} \ker(\chi) \subset \bigcap_{\bar{\nu} \in \psi(X_{\tilde{J}}(\tilde{\rho}))} \ker(\nu) = \psi\Big(\bigcap_{\nu \in X_{\tilde{J}}(\tilde{\rho})} \ker(\nu)\Big) = {}^\dagger J\, {}^\flat G.$$

Using (3.5) we know that $[\mathcal{I}_G(\rho){}^\flat G : \bigcap_{\chi \in X_G(\pi)} \ker(\chi)] = [\mathcal{I}_G(\rho){}^\flat G : {}^\dagger J\, {}^\flat G]$. Therefore, $\bigcap_{\chi \in X_G(\pi)} \ker(\chi)$ and ${}^\flat G {}^\dagger J$ must be equal and also ${}^\dagger J = \mathcal{I}_G(\rho) \cap \bigcap_{\chi \in X_G(\pi)} \ker(\chi)$.

For the last statement observe that $\mathrm{ind}_{{}^\dagger J}^{{}^\dagger J\, {}^\flat G}({}^\dagger \rho)$ is irreducible (since $\mathcal{I}_G({}^\dagger \rho) \subset \mathcal{I}_G(\rho)$), occurs in $\mathrm{Res}^G_{{}^\dagger J\, {}^\flat G}(\pi)$ and satisfies $\mathrm{Res}^{{}^\dagger J G}_G(\mathrm{ind}_{{}^\dagger J}^{{}^\dagger J\, {}^\flat G}({}^\dagger \rho)) = \mathrm{ind}_J^{{}^\flat G}(\rho) = {}^\flat \pi$. We now conclude using (4) Lemma 2.2. □

3.3. Center of Hecke algebras.

The following result describes the center of the Hecke algebra of a supercuspidal type.

Theorem 3.13. *Let $(J, \rho) \in \mathcal{J}_G^{wc}$. We have the following isomorphisms of $\mathbb{C}$-algebras*

$$\mathcal{Z}(\mathcal{H}(G, \rho)) = \mathcal{Z}(\mathcal{H}(\mathcal{I}_G(\rho), \rho)) = \mathcal{H}({}^\dagger J, \rho) \simeq \mathbb{C}[{}^\dagger J/J].$$

The two first are canonical, while the last is not.

Proof. The first equality follows readily from Lemma 3.8. Given Lemma 2.2, the proof of [21, Proposition 1.6.3.2] shows (upon replacing ${}^0 G$ in *loc. cit.* by ${}^\flat G$, which amounts to replacing ${}^0\pi$ in *loc. cit.* by ${}^\flat \pi$) that

$$\mathcal{Z}(\mathcal{H}(G, {}^\flat \pi)) = \mathcal{H}({}^\dagger({}^\flat G), {}^\flat \pi),$$

where ${}^\dagger({}^\flat G) = \bigcap_{\chi \in X_G(\pi)} \ker(\chi) = {}^\dagger J\, {}^\flat G$. Lemma 3.8 applied to ${}^\dagger({}^\flat G)$ shows that

$$\mathcal{H}({}^\dagger({}^\flat G), {}^\flat \pi) = \mathcal{H}(\mathcal{I}_{{}^\dagger({}^\flat G)}(\rho), \rho) = \mathcal{H}({}^\dagger({}^\flat G) \cap \mathcal{I}_G(\rho), \rho).$$

Now applying Lemma 3.12 we get ${}^\flat G) \cap \mathcal{I}_G(\rho) = {}^\dagger J$ and so $\mathcal{Z}(\mathcal{H}(G, {}^\flat \pi)) = \mathcal{H}({}^\dagger J, \rho)$.

We could have also reproduced the same argument of [21, Proposition 1.6.3.2] with ${}^\dagger J$ playing the role of ${}^\dagger({}^\flat G)$ and $\mathcal{I}_G(\rho)$ that of G and prove directly the second isomorphism above.

For the last isomorphism, we have the isomorphism

$$\mu: \mathbb{C}[{}^\dagger J/J] \xrightarrow{\sim} \mathrm{End}_{{}^\dagger J}(\rho \otimes \mathbb{C}[{}^\dagger J/J]) = \mathcal{H}({}^\dagger J, \rho)$$

defined in the proof of Lemma 3.9. This concludes the proof of the theorem. □

3.4. Criterion for the Hecke algebra of a supercuspidal type to be commutative.

Proposition 3.14. *Let $(J, \rho) \in \mathcal{J}_G^{wc}$. The following statements are equivalent:*

(1) The representation $\mathrm{Res}^G_{{}^\flat G}(\pi)$ is multiplicity free.

(2) The representation $\mathrm{Res}^{\tilde{J}}_J(\tilde{\rho})$ is also multiplicity free.

(3) The Hecke algebra $\mathcal{H}(G, \rho)$ is commutative.

Proof. Given that ${}^\flat G$ is open we know that $\mathcal{H}(G, {}^\flat \pi) \simeq \mathcal{H}(G, \rho) \simeq \mathcal{H}(\mathcal{I}_G(\rho), \rho)$ thanks to the transitivity of the induction. So by Theorem 3.13 and Lemma 3.9 we have the equivalence (2) $\Leftrightarrow$ (3) and Proposition 3.10 gives (1) $\Leftrightarrow$ (2). □

4. Weyl action on (center of) Hecke algebras and a Satake isomorphism

The results in this section are generalizations of [11, Section 1.6 - Section 1.9], where similar results are obtained for cuspidal types.

4.1. G-equivalence of types.

4.1.1. For $(J,\rho) \in \mathcal{J}_G^{wc}$, let $[\rho]$ be the set of irreducible representations of G that contain ρ and $[\pi\bar{\nu}]_G$ be the subset of irreducible representations of G whose inertial support contains $\pi \otimes \bar{\nu}$ for $\nu \in X({}^{\natural}J/J)$.

Definition 4.1. We say that two types (J,ρ) and (J',ρ') are G-equivalent and write it as $(J,\rho) \cong_G (J',\rho')$ if $\mathrm{ind}_J^G(\rho) \simeq \mathrm{ind}_{J'}^G(\rho')$.

Lemma 4.2. *Let (J,ρ) and (J',ρ') be two pairs from $\mathcal{J}_G$. The following properties are equivalent:*

(1) $[\rho] \cap [\rho'] \neq \varnothing$;
(1)' $(J,\rho) \approx_G (J',\rho')$;
(2) $[\rho] = [\rho']$;
(3) $\mathrm{Hom}_G(\mathrm{ind}_{J'}^G(\rho'), \mathrm{ind}_J^G(\rho)) \neq 0$.
If $(J,\rho), (J',\rho') \in \mathcal{J}_G^{wc}$, this is also equivalent to
(4) $(J,\rho) \cong_G (J',\rho')$.

Proof. (1) $\Leftrightarrow$ (2) is clear since any two orbits in $\mathrm{Irr}(G)$ under the action of $X(G/{}^{\flat}G)$ are disjoint or equal and by Theorem 3.2 we have

$$[\rho] = \bigsqcup_{\nu \in X({}^{\natural}J/J)} [\pi\bar{\nu}]_G \text{ and } [\rho'] = \bigsqcup_{\nu \in X({}^{\dagger}J'^{\circ}/J')} [\pi'\bar{\nu}]_G.$$

(1)$'$ $\Leftrightarrow$ (2) follows from [5, Proposition 3.5].

(2) $\Rightarrow$ (3). Let I be a system of representatives of ${}^{\dagger}J\backslash G/J'$. Using Mackey formula and Frobenius reciprocity we have

$$\begin{aligned}
\mathrm{Hom}_G(\mathrm{ind}_{J'}^G(\rho'), \mathrm{ind}_J^G(\rho)) &\simeq \mathrm{Hom}_G(\mathrm{ind}_{J'}^G(\rho'), \mathrm{ind}_J^G(\mathrm{Res}_J^{{}^{\dagger}J}({}^{\dagger}\rho)) \\
&\simeq \mathrm{Hom}_G(\mathrm{ind}_{J'}^G(\rho'), \mathrm{ind}_{{}^{\dagger}J}^G({}^{\dagger}\rho \otimes_{\mathbb{C}} \mathbb{C}[{}^{\dagger}J/J])) \\
&\simeq \bigoplus_{x \in I} \mathrm{Hom}_{J' \cap {}^{\dagger}J^x}(\rho', ({}^{\dagger}\rho \otimes_{\mathbb{C}} \mathbb{C}[{}^{\dagger}J/J])^x) \\
&\simeq \bigoplus_{x \in I} \mathrm{Hom}_{J' \cap {}^{\dagger}J^x}\left(\rho', {}^{\dagger}\rho^x \otimes_{\mathbb{C}} \mathbb{C}[({}^{\dagger}J)^x/J^x]\right) \\
&\simeq \bigoplus_{x \in I} \mathrm{Hom}_{J' \cap J^x}(\rho', {}^{\dagger}\rho^x) \otimes_{\mathbb{C}} \mathbb{C}[({}^{\dagger}J)^x/J^x] \\
&\simeq \bigoplus_{x \in I} \mathrm{Hom}_{J' \cap J^x}(\rho', {}^{\dagger}\rho^x) \otimes_{\mathbb{C}} \mathbb{C}[{}^{\dagger}J^x/J^x] \\
&\simeq \mathrm{Hom}_G(\mathrm{ind}_{J'}^G(\rho'), \mathrm{ind}_{{}^{\dagger}J}^G({}^{\dagger}\rho)) \otimes_{\mathbb{C}} \mathbb{C}[{}^{\dagger}J/J].
\end{aligned}$$

The commutation of the tensor product in the fifth equation comes from the fact that $J' \cap ({}^{\dagger}J)^x = J' \cap J^x$, hence $J' \cap {}^{\dagger}J^x$ acts trivially on $\mathbb{C}[{}^{\dagger}J^x/J^x]$. The isomorphism from the sixth to seventh equation follows from Mackey formula and is obtained as follows. Let $\psi \in \mathrm{Hom}_G(\mathrm{ind}_{J'}^G(\rho'), \mathrm{ind}_{{}^{\dagger}J}^G({}^{\dagger}\rho))$ map to $(\psi_i)_{i \in I} \in (\bigoplus_{i \in I} \mathrm{Hom}_{J' \cap {}^{\dagger}J^x}(\rho', {}^{\dagger}\rho^x))$. The isomorphism is then given by sending $\psi \otimes \chi \to (\psi_i \otimes \chi^x)_{i \in I}$.

Now by Lemma 2.2 we deduce

$$\mathrm{Hom}_G(\mathrm{ind}_{J'}^G(\rho'), \mathrm{ind}_J^G(\rho)) \simeq m_{{}^\flat G}(\pi)\,\mathrm{Hom}_G(\mathrm{ind}_{J'}^G(\rho'), \pi) \otimes \mathbb{C}[{}^\dagger J/J]. \tag{4.1}$$

By Theorem 3.2, the pair (J', ρ') is a type for G. Therefore

$$\mathrm{Hom}_G(\mathrm{ind}_{J'}^G(\rho'), \mathrm{ind}_J^G(\rho)) \neq 0 \Leftrightarrow \pi \in [\rho'].$$

(4) $\Rightarrow$ (1) is clear, let us show (1)$'$ $\Rightarrow$ (4) assuming $\mathcal{I}_G(\rho) \subset \widetilde{J}$: By Theorem 3.2, we have an equality $[G, \pi']_G = [G, \pi \otimes \bar{\nu}]_G$ for some $\nu \in X({}^\natural J/J)$, and this is equivalent to (by Corollary 3.3) $\pi \simeq \pi' \otimes \chi$ for some $\chi \in X(G/{}^\flat G)$, which is equivalent to (by (5) Lemma 2.1) $\mathrm{ind}_{J'}^{{}^\flat G}(\rho') \simeq (\mathrm{ind}_J^{{}^\flat G}(\rho))^x \simeq \mathrm{ind}_{J^x}^{{}^\flat G}(\rho^x)$ for some $x \in G$, since $\mathrm{ind}_J^{{}^\flat G}(\rho) \in \mathcal{O}_{{}^\flat G}(\pi)$ and $\mathrm{ind}_{J'}^{{}^\flat G}(\rho') \in \mathcal{O}_{{}^\flat G}(\pi')$. This latter implies $\mathrm{ind}_{J'}^G(\rho') \simeq (\mathrm{ind}_J^G(\rho))^x \simeq \mathrm{ind}_J^G(\rho)$. □

Corollary 4.3. *Let $(J, \rho) \in \mathcal{J}_G$. We have an isomorphism of $\mathbb{C}$-algebras*

$$\mathcal{H}(G, \rho) \simeq \bigoplus_{\nu \in X({}^\natural J/J)} \mathcal{H}(G, {}^\natural\rho \otimes \nu).$$

Proof. We first note the isomorphism of $\mathbb{C}$-modules

$$\mathrm{End}_G(\mathrm{ind}_J^G(\rho)) = \bigoplus_{\nu,\nu' \in X({}^\natural J/J)} \mathrm{Hom}_G(\mathrm{ind}_{{}^\natural J}^G({}^\natural\rho \otimes \nu), \mathrm{ind}_{{}^\natural J}^G({}^\natural\rho \otimes \nu')).$$

Since $[{}^\natural\rho \otimes \nu'] = [{}^\natural\rho \otimes \nu]$ if and only if $\nu = \nu'$, Lemma 4.2 yields an isomorphism of $\mathbb{C}$-modules

$$\mathrm{End}_G(\mathrm{ind}_J^G(\rho)) = \bigoplus_{\nu \in X({}^\natural J/J)} \mathrm{End}_G(\mathrm{ind}_{{}^\natural J}^G({}^\natural\rho \otimes \nu)).$$

which is clearly an isomorphism of $\mathbb{C}$-algebras. This concludes the proof. □

Corollary 4.4. *If $(J, \rho) \in \mathcal{J}_G^{wc}$ then $m_J({}^0\rho) = 1$, i.e. $({}^\natural J, {}^\natural\rho) = ({}^0 J, {}^0\rho)$ and*

$$\mathcal{H}(G, \rho) \simeq \bigoplus_{\nu \in X({}^0 J/J)} \mathcal{H}(G, {}^0\rho \otimes {}^0\nu).$$

In particular, $\mathcal{H}(G, \rho)$ is commutative if and only if $\mathcal{H}(G, {}^0\rho)$ is commutative.

Proof. As in Lemma 3.5, we have $\mathcal{I}_G({}^0\rho) \subset {}^0 J \mathcal{I}_G(\rho) {}^0 J$ and $\mathcal{I}_G({}^\natural\rho) \subset \mathcal{I}_G(\rho)$, hence

$$\mathcal{I}_G(\rho) \subset \widetilde{J} \Rightarrow \mathcal{I}_G({}^\natural\rho) \subset \widetilde{J} \text{ and } \mathcal{I}_G({}^0\rho) \subset \widetilde{J}.$$

This shows that the types $({}^0 J, {}^0\rho \otimes {}^0\nu)$ and $({}^\natural J, {}^\natural\rho \otimes \nu)$ are both in $\mathcal{J}_G^{wc}$. In (2) Theorem 3.2 we saw that $({}^0 J, {}^0\rho \otimes {}^0\nu) \approx_G ({}^\natural J, {}^\natural\rho \otimes \nu)$, which is equivalent by Lemma 4.2 to $\mathrm{ind}_{{}^\natural J}^G({}^\natural\rho \otimes \nu) \simeq \mathrm{ind}_{{}^0 J}^G({}^0\rho \otimes {}^0\nu)$ and so $m_J({}^0\rho) = 1$. Accordingly, Corollary 4.3 yields

$$\mathcal{H}(G, \rho) \simeq \bigoplus_{\nu \in X({}^0 J/J)} \mathcal{H}(G, {}^0\rho \otimes {}^0\nu).$$

The last claim, follows from the fact that $m_J(\tilde{\rho}) = m_J({}^0\rho) \cdot m_{{}^0 J}(\tilde{\rho}) = m_{{}^0 J}(\tilde{\rho})$ and Proposition 3.14. □

This shows explicitly how we move from the case ${}^\flat G$ to ${}^0 G$ and vice versa.

4.2. **Normalizer of a type.** Let $\mathbf{M}$ be a Levi subgroup of $\mathbf{G}$ and ${}^\flat M \subset M \cap {}^\flat G$. This an open subgroup of M of finite index in ${}^0M$ and containing M^{der}. We write $\mathcal{J}_M$ (resp. $\mathcal{J}_M^{wc}$) as in §3. For $(J_M, \rho_M) \in \mathcal{J}_M$ we will regularly use the notation $\pi_M := \mathrm{ind}_{\tilde{J}_M}^M(\tilde{\rho}_M) \in \mathrm{Irr}(M)$.

The normalizer $N_G(M)$ of a Levi M acts naturally by conjugation on $\mathrm{Irr}(M)$. Recall that σ^n denotes the conjugate of any $\sigma \in \mathrm{Irr}(M)$ by an element $n \in N_G(M)$ and the pair (J_M^n, ρ_M^n) the conjugate of (J_M, ρ_M). Moreover, $\mathcal{I}_M(\rho_M^n) = (\mathcal{I}_M(\rho_M))^n$ is a group such that $\mathcal{H}(M, \rho_M) \simeq \mathcal{H}(M, n(\rho_M))$.

Proposition 4.5. *Let (J, ρ) be a G-cover (*[5, Definition 8.1]*) of $(J_M, \rho_M) \in \mathcal{J}_M$. For any $n \in N_G(M)$, the following statements are equivalent:*

(1) $n[\rho_M] \cap [\rho_M] \neq \varnothing$;

(1)' $n[\rho_M] = [\rho_M]$;

(2) $\mathrm{Hom}_M(n(\mathrm{ind}_{J_M}^M(\rho_M)), \mathrm{ind}_{J_M}^M(\rho_M)) \neq 0$;

(3) there exists $m \in M$ such that $mn \in \mathcal{I}_G(\rho)$.
If $(J_M, \rho_M), \in \mathcal{J}_M^{wc}$, this is also equivalent to

(4) $(J_M^n, \rho_M^n) \cong_M (J_M, \rho_M)$.

The group $N_G(\rho_M) := \{n \in N_G(M) \colon (J_M^w, \rho_M^w) \cong_M (J_M, \rho_M)\}$ is called the normalizer of the type (J_M, ρ_M). In particular, Given (1)', the stabilizer of $\mathfrak{S}(J_M, \rho_M)$ in the Weyl group is $W_{[\rho_M]} := N_G(\rho_M)/M$.

Proof. As in [11, Proposition 1.9.1], observe that $n(\mathrm{ind}_{J_M}^M(\rho_M)) \simeq (\mathrm{ind}_{J_M^n}^M(\rho_M^n))$. The equivalences (1) ⇔ (1)′ ⇔ (2) ⇔ (4), follow then from Lemma 4.2.

(2) ⇔ (3) As we saw in the proof of Lemma 4.2 we have

$$(2) \Leftrightarrow \mathrm{Hom}_M(\mathrm{ind}_{J_M}^M(\rho_M), \pi_M^n) \neq 0.$$

Using Frobenius reciprocity and then Mackey formula we see that the right hand side is equivalent to

$$\mathrm{Hom}_{J_M \cap \tilde{J}_M^{m'n}}(\mathrm{Res}^{J_M}_{J_M \cap \tilde{J}_M^{m'n}}(\rho_M), \mathrm{Res}^{\tilde{J}_M^{m'n}}_{J_M \cap \tilde{J}_M^{m'n}}(\tilde{\rho}_M^{m'n})) \neq 0 \quad \text{for some } m' \in M.$$

Now since $J_M \cap \tilde{J}_M^{m'n} = J_M \cap J_M^{m'n}$ we deduce (using Clifford theory) that the previous statement is equivalent to

$$\mathrm{Hom}_{J_M \cap J_M^{mn}}(\mathrm{Res}^{J_M}_{J_M \cap J_M^{mn}}(\rho_M), \mathrm{Res}^{J_M^{mn}}_{J_M \cap J_M^{mn}}(\rho_M^{mn})) \neq 0 \quad \text{for some } m \in M.$$

Let P be any parabolic subgroup with Levi factor M and a radical unipotent U, $\bar{U}$ its opposite. By definition of a cover, we have an Iwahori decomposition for J with respect to any parabolic subgroup with Levi component M. Now it suffices to observe that

$$J \cap J^{mn} = (J \cap J^{mn} \cap U) \cdot (J_M \cap J_M^{mn}) \cdot (J \cap J^{mn} \cap \bar{U})$$

and that ρ and ρ^{mn} are both trivial on both unipotent factors $J \cap J^{mn} \cap U$ and $J \cap J^{mn} \cap \bar{U}$. Therefore,

$$\mathrm{Hom}_{J \cap J^{mn}}(\mathrm{Res}^J_{J \cap J^{mn}}(\rho), \mathrm{Res}^{J^{mn}}_{J \cap J^{mn}}(\rho^{mn})) = \mathrm{Hom}_{J_M \cap J_M^{mn}}(\mathrm{Res}^{J_M}_{J_M \cap J_M^{mn}}(\rho_M), \mathrm{Res}^{J_M^{mn}}_{J_M \cap J_M^{mn}}(\rho_M^{mn})).$$

Which shows $mn \in \mathcal{I}_G(\rho) \Leftrightarrow (2)$. This concludes the proof of the proposition. □

Remark 4.6. As observed in [11, Remarque §1.9]: Let $(J_M, \rho_M) \in \mathcal{J}_M^{wc}$. If $n \in N_{[\rho_M]}$, then $\pi_M^n \simeq \pi_M \otimes \chi_n$ for some $\chi_n \in X(M/{}^\flat M)$. We know that $\pi_M \simeq \pi_M \otimes \chi$ if and only if $\chi \in X_M(\pi_M) = X(M/{}^\dagger J_M{}^\flat M)$ (Lemma 3.12). Hence, for any character $\chi \in X(M/{}^\dagger J_M{}^\flat M)$ we have $\pi_M \otimes \chi_n \simeq \pi_M^n \simeq (\pi_M \otimes \chi)^n \simeq \pi_M \otimes \chi_n \chi^n$, which shows that $\pi_M \simeq \pi_M \otimes \chi^n$ i.e. χ^n still lives in $X_M(\pi_M)$. This shows that $N_{[\rho_M]}$ normalizes ${}^\dagger J_M{}^\flat M$. Accordingly, $N_{[\rho_M]}$ defines an action (by conjugation) on $X({}^\dagger J_M{}^\flat M/{}^\flat M) = X({}^\dagger J_M/J_M)$.

Corollary 4.7. *For any two G-covers (J,ρ) and (J',ρ') of two types $(J_M,\rho_M) \in \mathcal{J}_M^{wc}$ and $(J'_M,\rho'_M) \in \mathcal{J}_M^{wc}$, the following properties are equivalent:*
(1) $[\rho] \cap [\rho'] \neq \varnothing$*;*
(2) $(J,\rho) \cong_G (J',\rho')$.

Proof. Given Proposition 4.5 this is the same as [11, Proposition 4.5.1]. We remark that Proposition 4.5.1 of *loc. cit.* assumes Conjecture 1.4 in *loc. cit.*, which is verified is verified in Section 1.5 *loc. cit.* in the complex case. See also [1, Lemma B.3]. □

4.3. **Weyl action on the center.**

4.3.1. Let $(J_M,\rho_M) \in \mathcal{J}_M^{wc}$. An element $z \in \mathfrak{Z}^{[\rho_M]}$ is a collection of morphisms $z_\sigma \in \mathrm{End}_M(\sigma), \forall \sigma \in \mathrm{Obj}(\mathfrak{R}_{\rho_M})$, such that $f \circ z_\sigma = z_\tau \circ f$ for any morphism $f \in \mathrm{Hom}_M(\sigma,\tau), \forall \sigma,\tau \in \mathfrak{R}_{\rho_M}$. In particular, $z_\sigma \in \mathcal{Z}(\mathrm{End}_M(\sigma)), \forall \sigma \in \mathrm{Obj}(\mathfrak{R}_{\rho_M})$. One case of interest: If $\Sigma = \mathrm{ind}_{J_M}^M(\rho_M)$ then $z_\Sigma \in \mathcal{Z}(\mathcal{H}(M,\rho_M))$.

The equivalence of categories (Definition 1.1) $\mathfrak{M}_{\rho_M}$ induces a ring isomorphism

$$m_{\rho_M} \colon \mathfrak{Z}^{[\rho_M]} \xrightarrow{\sim} \mathcal{Z}(\mathcal{H}(M,\rho_M)), \quad z = (z_\sigma)_{\sigma \in \mathfrak{R}_{\rho_M}} \longmapsto z_{\mathrm{ind}_{J_M}^M(\rho_M)}.$$

Let (J,ρ) be a G-cover of (J_M,ρ_M). We know by [5, Theorem 8.3] that (J,ρ) is a $\mathfrak{S}(J,\rho)$-type and Theorem 3.2 gives an explicit description for this set:

$$\mathfrak{S}(J,\rho) = \{[M, \sigma \otimes \bar{\nu}]_G \colon \nu \in X({}^\natural J_M/J_M)\}.$$

Therefore, we also have an isomorphism of rings

$$m_\rho \colon \mathfrak{Z}^{[\rho]} \xrightarrow{\sim} \mathcal{Z}(\mathcal{H}(G,\rho)), \quad z = (z_\sigma)_{\sigma \in \mathfrak{R}_\rho} \longmapsto z_{\mathrm{ind}_J^G(\rho)}.$$

4.3.2. In this section, we define an action of $N_G(\rho_M)$ on $\mathfrak{Z}^{[\rho_M]}$ and by transport of structure we get an action on $\mathcal{Z}(\mathcal{H}(M,\rho_M))$ that is compatible with the isomorphism m_{ρ_M}.

Let $n \in N_G(M)$, Proposition 4.5 shows that n normalizes $[\rho_M]$ if and only if $(\mathrm{ind}_{J_M}^M(\rho_M))^n \simeq \mathrm{ind}_{J_M}^M(\rho_M)$. Accordingly, for any $n \in N_G(\rho_M)$ and any $(\sigma,\mathcal{V}) \in \mathrm{Obj}(\mathfrak{R}_{\rho_M})$ we clearly have $\sigma^n \in \mathrm{Obj}(\mathfrak{R}_{\rho_M})$.

- For any $n \in N_G(\rho_M)$ and $z \in \mathfrak{Z}^{[\rho_M]}$, define the following map:

$$n \cdot z = ((n \cdot z)_\sigma \coloneqq z_{\sigma^{n^{-1}}})_{\sigma \in \mathfrak{R}_{\rho_M}}.$$

This defines an action of $N_G(\rho_M)$ on $\mathfrak{Z}^{[\rho_M]}$. Now if $m \in M$, then we have a commutative diagram

$$\begin{array}{ccc} \sigma & \xrightarrow{r_m} & \sigma^{m^{-1}} \\ {\scriptstyle z_\sigma}\downarrow & & \downarrow{\scriptstyle z_{\sigma^{m^{-1}}}} \\ \sigma & \xrightarrow{r_m} & \sigma^{m^{-1}} \end{array}$$

where r_m is the isomorphism given by $v \mapsto \sigma(m^{-1})(v)$. It follows readily that $z_\sigma = z_{\sigma^{m^{-1}}}$. Accordingly, the defined action of $N_G(\rho_M)$ factors through $W_{[\rho_M]}$.

- Write $\mathcal{W}$ for the underlying space of $\Sigma = \mathrm{ind}_{J_M}^M(\rho_M)$. For any $n \in N_G(\rho_M)$ choose an element $\bar{w} \in \mathrm{Aut}_{\mathbb{C}}(\mathcal{W})$ which realizes the isomorphism $\Sigma \xrightarrow{\sim} \Sigma^{n^{-1}}$. Given an element $z \in \mathfrak{Z}(\mathfrak{R}_{\rho_M})$, the following diagram is by definition commutative.

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{\bar{w}} & \mathcal{W} \\ {\scriptstyle z_\Sigma}\downarrow & & \downarrow{\scriptstyle z_{\Sigma^{n^{-1}}}} \\ \mathcal{W} & \xrightarrow{\bar{w}} & \mathcal{W} \end{array}$$

Thus $m_{\rho_M}(n \cdot z) = (n \cdot z)_\Sigma = \bar{w} \circ z_\Sigma \circ \bar{w}^{-1} = \bar{w} \circ m_{\rho_M}(z) \circ \bar{w}^{-1}$. So by transport of structure we get the following action on the center of the Hecke algebra:

$$n \cdot \psi = \bar{w} \circ \psi \circ \bar{w}^{-1}, \quad \forall n \in W_{[\rho_M]}, \forall \psi \in \mathcal{Z}(\mathcal{H}_M(M, \rho_M)).$$

Finally, we fix the isomorphism $\mu_M : \mathbb{C}[{}^\dagger J_M / J_M] \to \mathcal{Z}(\mathcal{H}(M, \rho_M))$ as in Theorem 3.13 (which is not canonical). We use this isomorphism to give an action of $W_{[\rho_M]}$ on $\mathbb{C}[{}^\dagger J_M / J_M]$ by transport of structure.

Theorem 4.8. *Assume ${}^\flat M = {}^0 M$. Let $(J_M, \rho_M) \in \mathcal{J}_M^{wc}$ and (J, ρ) a G-cover. We have the following isomorphism of $\mathbb{C}$-algebras*

$$\mathcal{Z}(\mathcal{H}(G, \rho)) = \mathcal{Z}(\mathcal{H}(M, \rho_M))^{W_{[\rho_M]}} \simeq \mathbb{C}[{}^\dagger J_M / J_M]^{W_{[\rho_M]}},$$

where, the first is canonical while the second is not.

Proof. By [21, Theorem 1.9.1.1] we know that

$$\mathfrak{Z}^{\mathfrak{s}} = (\mathfrak{Z}^{\mathfrak{t}})^{W_{\mathfrak{t}}},$$

where, $\mathfrak{t} := [M, \pi_M]_M = \mathfrak{S}(J_M, \rho_M)$, $\mathfrak{s} := [M, \pi_M]_G$ and $W_{\mathfrak{t}} = W_{[\rho_M]}$.

By $W_{[\rho_M]}$-equivariance of $m_{[\rho_M]}$, we get a canonical isomorphism

$$\mathcal{Z}(\mathcal{H}(G, \rho)) = (\mathcal{Z}(\mathcal{H}(M, \rho_M)))^{W_{[\rho_M]}}.$$

And finally, we conclude using the $W_{[\rho_M]}$-equivariance of μ_M. □

Remark 4.9. Following Roche [20, Remark 1.6.4.2], choose a support preserving isomorphism of $\mathbb{C}$-algebras $n_{[\rho_M]} : \mathbb{C}[{}^\dagger J_M / J_M] \xrightarrow{\sim} \mathfrak{Z}^{[\rho_M]}$ constructed as follows: For each $x \in {}^\dagger J_M$, we write z^x for the element of $\mathfrak{Z}^{[\rho_M]}$ with support ${}^0 M x^{-1}$ such that $z^x(gx^{-1}) = {}^\dagger({}^0\pi_M)(gx^{-1})$, for $g \in {}^0 M$. As we saw in Remark 4.6, $\pi_M^n \simeq \pi_M \otimes \chi_n$ for some $\chi_n \in X(M/{}^0 M)$. For any $n \in W_{[\rho_M]}$ and $\sigma = \pi_M \otimes \chi \in [\rho_M]$ we have that

$$(n \cdot \mathfrak{n}_{\rho_M}(x))_\sigma = z^x_{\sigma^{n^{-1}}} = z^x_{\pi_M \otimes \chi_n \chi^{n^{-1}}}$$

which, as explained in *loc.cit.*, acts on the underlying space by $\chi_n(x)\chi(nxn^{-1})$. On the other hand,

$$(\mathfrak{n}_{\rho_M}(nxn^{-1}))_\sigma = z^{nxn^{-1}}_{\pi_M \otimes \chi}$$

which acts on the underlying space by $\chi(nxn^{-1})$. Hence, with natural conjugation action of $W_{[\rho_M]}$ on $\mathbb{C}[{}^\dagger J_M / J_M]$ as in Remark 4.6, this morphism n_{ρ_M} is not $W_{[\rho_M]}$-equivariant.

Example 4.10. Assume that $\mathbf{M_0}$ is a minimal Levi and that $({}^0M_0, 1_{{}^0M_0})$ is its maximal open compact subgroup together with its trivial character. This is a cuspidal type for $\mathfrak{t} = [M_0, 1]$ and $W^{\mathfrak{t}} = W_0 = N_G(M_0)/M_0$. Clearly $M_0 = \mathcal{I}_{M_0}(1) = {}^{\dagger}M_0$. Let I be an (maximal) Iwahori open compact subgroup containing ${}^0M_0$. The pair $(I, 1_I)$ is a G-cover for $({}^0M_0, 1)$. So Theorem 4.8 shows in this particular case the classical Satake isomorphism

$$\mathcal{Z}(\mathcal{H}(G, 1_I)) \simeq \mathbb{C}[M_0/{}^0M_0]^{W_0}.$$

5. Some nice families of compact open subgroups

Let K be a compact open subgroup of G and let $\mathfrak{R}_K(G)$ be the full sub-category of $\mathfrak{R}(G)$ consisting of representations (π, V) that are generated by their K-fixed vectors. Write $\mathcal{H}(G,K)$ for the Hecke algebra $\mathcal{H}(G, 1_K)$, where 1_K denotes the trivial representation of K.

Let $\mathbf{S}$ be a maximal split torus in $\mathbf{G}$. In [2, Section 3.7] the authors introduce criteria on K, which we call $\heartsuit_S$ and recall now.

Definition 5.1. Let K be a compact open subgroup of G. We say K satisfies $\heartsuit_S$ if

(1) Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$ that contains $\mathbf{S}$. Write $\mathbf{P} = \mathbf{MN}$ with $\mathbf{M}$. Let K' be a G-conjugate of K and let $K'_P = K' \cap P / K' \cap N$. For any parabolic subgroup $\mathbf{Q}$ of $\mathbf{G}$ with the same Levi subgroup $\mathbf{M}$ and any other G-conjugate K_1 of K, $(K_1)_Q$ is a conjugate of K'_P in M.
(2) Let (σ, V) be a representation of G. Let $V(N) = Span\langle \sigma(n)v - v | v \in V, n \in N\rangle$ and let $V_N = V/V(N)$. Let V^K be the set of K-fixed vectors of V. Then the canonical map $V^K \to V_N^{M\cap K}$ is surjective.

Let $\mathcal{K}^{\heartsuit}(S,G)$ be the collection of all compact open subgroups of G that satisfies $\heartsuit_S$. Let us recall the following proposition.

Proposition 5.2 (Corollary 3.9 of [2])**.** *Let* $\mathbf{S}$ *be a maximal* F*-split torus in* G *and let* $K \in \mathcal{K}^{\heartsuit}(S,G)$. *The pair* $(K,1)$ *is* $\mathfrak{S}(K)$*-type for a finite set* $\mathfrak{S}(K) := \mathfrak{S}(K,1) \subset \mathfrak{B}(G)$ *(see Definition 1.1).*

Lemma 5.3. *Let* $K \in \mathcal{K}^{\heartsuit}(S,G)$. *We have* $\mathfrak{s} = [M,\sigma]_G \in \mathfrak{S}(K)$ *if and only if* $\sigma^{K\cap M} \neq 0$.

Proof. The proof given in [7, Proposition 4] goes through verbatim. □

5.1. Some compact open subgroups that live in $\mathcal{K}^{\heartsuit}(S,G)$.

In [3, Section 5], the following condition is considered in place of $\heartsuit_S$ above.

Definition 5.4. Let $\mathbf{S}$ be a maximal F-split torus in $\mathbf{G}$. Let K be a compact open subgroup of G and let K^G be the set of G-conjugates of K. We say K satisfies $\spadesuit_S$ if, for any parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ that contains $\mathbf{S}$, any P-conjugacy class of K^G contains a K' that admits an Iwahori decomposition with respect to P:

$$K' = (K' \cap N^-)(K' \cap M)(K' \cap N).$$

Let $\mathcal{K}^{\spadesuit}(S,G)$ be the collection of compact open subgroups of G that satisfy $\spadesuit_S$. It is shown in [3, Proposition 5.1] that Proposition 5.2 holds for all $K \in \mathcal{K}^{\spadesuit}(S,G)$.

Let $\mathcal{B}(G,F)$ (resp. $\mathcal{B}(\mathbf{G}_{\breve{F}},\breve{F})$) denote the Bruhat-Tits building of $\mathbf{G}$ over F (resp. $\mathbf{G}_{\breve{F}}$ over $\breve{F}$). Let $\mathcal{A}(S,F)$ denote the apartment of $\mathbf{S}$ over F. For $r \in \mathbb{R}_{\geq 0}$ and $x \in \mathcal{B}(G,F)$ let $G_{x,r}$ denote the Moy-Prasad filtration subgroup (see [17, 18]). By [3, Proposition 5.2], we have $G_{x,r} \in \mathcal{K}^{\spadesuit}(S,G)$ for all $x \in \mathcal{A}(S,F)$ and $r > 0$.

We are interested in compact open subgroups for which Lemma 5.3 holds, that is in compact open subgroups that lie in $\mathcal{K}^{\heartsuit}(S,G)$. Before taking this up, let us recall some preliminaries about filtrations of root subgroups from [10, Chapter 4 and Chapter 5].

5.1.1. *Filtration of root subgroups.* Recall that we have fixed a valuation ω on F so that $\omega(F^\times) = \mathbb{Z}$. Let $\breve{F}$ be the maximal unramified extension of F contained in F_s. Let $\mathbf{G}$ be a connected, reductive group over F. Then by [23], $\mathbf{G}_{\breve{F}}$ is quasi-split. Let σ denote the Frobenius action on $\mathbf{G}_{\breve{F}}$ so that $\mathbf{G} = \mathbf{G}_{\breve{F}}^{\sigma}$. Let $\widetilde{F}$ be the smallest sub-extension of F_s over which $\mathbf{G}_{\breve{F}}$ splits. Let $\mathbf{S}$ be a maximal F-split torus in $\mathbf{G}$ and let $\breve{\mathbf{S}}$ be a maximal $\breve{F}$-split F-torus containing $\mathbf{S}$. Let $\mathbf{T} = Z_{\mathbf{G}}(\breve{\mathbf{S}})$. Then $\mathbf{T}$ is a maximal torus in $\mathbf{G}$. Let $\breve{\boldsymbol{a}}$ be a σ-stable alcove in the apartment $\mathcal{A}(\breve{S},\breve{F})$ and let $\boldsymbol{a} = \breve{\boldsymbol{a}}^{\sigma}$. Then $\boldsymbol{a}$ is an alcove in the apartment $\mathcal{A}(S,F)$. Choose a special vertex x_0 in the closure of $\boldsymbol{a}$. Let $\breve{\Phi} = \Phi(\mathbf{G}_{\breve{F}},\breve{\mathbf{S}})$ denote the set of roots of $\breve{\mathbf{S}}$ in $\mathbf{G}_{\breve{F}}$. Similarly we have $\Phi = \Phi(\mathbf{G},\mathbf{S})$. The choice of x_0 in the closure of $\boldsymbol{a}$ determines a set of simple roots $\breve{\Delta}$ of $\breve{\Phi}$ and Δ of Φ. Let $\breve{\Phi}^{\mathrm{red}}$ (resp. $\breve{\Phi}^{\mathrm{nd}}$) denote the set of reduced (resp. non-divisible) roots of $\breve{\Phi}$. We similarly have Φ^{red} and Φ^{nd}.

Let us first recall the definition of filtration of roots subgroups for $\breve{a} \in \breve{\Phi}$. We will then prove a lemma that describes the set of jumps. Let $\mathbf{U}_{\breve{a}}$ be the root subgroup attached to $\breve{a}$. There are two possibilities.

(1) Suppose $\breve{a} \in \breve{\Phi}^{\mathrm{red}}$ is such that $2\breve{a}$ is not a root. We fix a pinning $(L_{\breve{a}}, x_{\breve{a}})$ as in [10, Section 4.1.5 and Section 4.1.8]. Here $L_{\breve{a}} \hookrightarrow \widetilde{F}$ and $x_{\breve{a}} : \mathbf{U}_{\breve{a}} \xrightarrow{\simeq} \mathrm{Res}_{L_{\breve{a}}/\breve{F}}\mathbb{G}_a$ is an isomorphism. Let $e_{\breve{a}} = [L_{\breve{a}} : \breve{F}]$. Let $\Gamma'_{\breve{a}} = \omega(L_{\breve{a}}^\times) = \frac{1}{e_{\breve{a}}}\mathbb{Z}$. The set of affine roots with gradient $\breve{a}$ are of the form $\breve{a}+m$ for $m \in \Gamma'_{\breve{a}}$. For $m = \frac{k}{e_{\breve{a}}} \in \Gamma'_{\breve{a}}$, we have $\mathbf{U}_{\breve{a}}(\breve{F})_{x_0,m} = x_{\breve{a}}^{-1}(\mathfrak{p}_{L_{\breve{a}}}^k)$ and for any real number r let $\mathbf{U}_{\breve{a}}(\breve{F})_{x_0,r} = \mathbf{U}_{\breve{a}}(\breve{F})_{x_0,m}$ where $m = inf\{h \in \Gamma'_{\breve{a}} \mid h \geq r\}$.

(2) Suppose $\breve{a} \in \breve{\Phi}^{\mathrm{red}}$ is such that $2\breve{a}$ is a root. We fix a pinning $(L_{\breve{a}}, L_{2\breve{a}}, x_{\breve{a}})$ as in [10, Section 4.1.9]. Here $L_{\breve{a}}$ is a quadratic extension of $L_{2\breve{a}}$ with unique non-trivial automorphism τ, $H_0(L_{\breve{a}}, L_{2\breve{a}}) = \{(u,v) \in L_{\breve{a}} \times L_{\breve{a}} \mid v + \tau(v) = u\tau(u)\}$ with multiplication given by [10, Equation (4) of Section 4.1.9]. Let $H(L_{\breve{a}}, L_{2\breve{a}}) = \mathrm{Res}_{L_{2\breve{a}}/\breve{F}} H_0(L_{\breve{a}}, L_{2\breve{a}})$ and let $x_{\breve{a}} : \mathbf{U}_{\breve{a}} \xrightarrow{\simeq} H(L_{\breve{a}}, L_{2\breve{a}})$. Let $e_{\breve{a}} = [L_{\breve{a}} : F]$ and $e_{2\breve{a}} = [L_{2\breve{a}} : F]$. Note that $e_{\breve{a}} = 2e_{2\breve{a}}$. As in [10, Lemma 4.3.3], let $L_{\breve{a}} = L_{2\breve{a}}[t]$ where $t^2 - \alpha t + \beta = 0$. If $\alpha = 0$, set $\lambda = \frac{1}{2}$. If $\alpha \neq 0$, set $\lambda = t\alpha^{-1}$. Let $\Gamma'_{\breve{a}}$ be the value set attached to the root $\breve{a}$ as in [10, Section 4.2.20]. Note that $\omega(L_{\breve{a}}^\times) = \frac{1}{e_{\breve{a}}}\mathbb{Z}$. Then by [10, Section 4.3.4],

$$\Gamma'_{\breve{a}} = \begin{cases} \frac{1}{e_{\breve{a}}}\mathbb{Z} & \text{if } \alpha = 0 \\ \frac{1}{2e_{\breve{a}}} + \frac{1}{e_{\breve{a}}}\mathbb{Z} & \text{if } \alpha \neq 0. \end{cases} \tag{5.1}$$

Let $L_{\breve{a}}^0$ be the set of elements in $L_{\breve{a}}$ of trace 0. Then again by [10, Section 4.3.4], $\Gamma'_{2\breve{a}} = \omega(L_{\breve{a}}^0\backslash\{0\})$. Let $\gamma = -\frac{1}{2}\omega(\lambda)$. For $m \in \Gamma'_{\breve{a}}$, let (see [10, Section 4.3.5])

$$\mathbf{U}_{\breve{a}}(\breve{F})_{x_0,m} = \left\{x_{\breve{a}}(u,v) \in H(L_{\breve{a}}, L_{2\breve{a}}) \mid \omega(u) \geq m+\gamma,\ \omega(v-\lambda u\tau(u)) \geq 2m + \frac{1}{e_a}\right\}.$$

This definition is extended to $r \in \mathbb{R}$ as in [10, Section 4.3.8].

Lemma 5.5. *Let $\breve{a} \in \breve{\Phi}^{\mathrm{red}}$. Let $m \in \frac{1}{e_{\breve{a}}}\mathbb{Z}$. Let $r \in \mathbb{R}$ be such that $0 < r < \frac{1}{e_{\breve{a}}}$ if $2\breve{a}$ is not a root, and such that $0 < r < \frac{1}{2e_{\breve{a}}}$ if $2\breve{a}$ is a root. Then $\mathbf{U}_{\breve{a}}(\breve{F})_{x_0,m-r} = \mathbf{U}_{\breve{a}}(\breve{F})_{x_0,m}$ and $\mathbf{U}_{\breve{a}}(\breve{F})_{x_0,m+r} = \mathbf{U}_{\breve{a}}(\breve{F})_{x_0,m+\frac{1}{e_{\breve{a}}}}$.*

Proof. Write $m = \frac{k}{e_{\breve{a}}}$ for a suitable $k \in \mathbb{Z}$.

Let $\breve{a} \in (\breve{\Phi})^{\mathrm{red}}$ be such that $2\breve{a}$ is not a root. Then $\Gamma'_{\breve{a}} = \frac{1}{e_{\breve{a}}}\mathbb{Z}$. We have assumed that $0 < r < \frac{1}{e_{\breve{a}}}$. So $\mathbf{U}_{\breve{a}}(\breve{F})_{x_0,m-r} = \mathfrak{p}_{L_{\breve{a}}}^k = \mathbf{U}_{\breve{a}}(\breve{F})_{x_0,m}$ and $\mathbf{U}_{\breve{a}}(\breve{F})_{x_0,m+r} = \mathfrak{p}_{L_{\breve{a}}}^{k+1} = \mathbf{U}_{\breve{a}}(\breve{F})_{x_0,m+\frac{1}{e_{\breve{a}}}}$.

Next, let $\breve{a} \in \breve{\Phi}$ be such that $2\breve{a}$ is a root. We have a few cases.

(1) Suppose $\alpha = 0$. Then $\lambda = \frac{1}{2}$. We have two subcases.

- Suppose the residue characteristic of F is not 2. Then $\omega(\lambda) = 0$. We have assumed that $0 < r < \frac{1}{2e_{\breve{a}}}$. Now, $\mathbf{U}_{\breve{a}}(\breve{F})_{x_0,m} = \{x_{\breve{a}}(u,v) \in H(L_{\breve{a}}, L_{2\breve{a}}) \mid \omega(u) \geq \frac{k}{e_{\breve{a}}},\ \omega(v-\lambda u\tau(u)) \geq \frac{2k}{e_{\breve{a}}} + \frac{1}{e_{\breve{a}}}\}$. Using the facts that
$$\frac{k}{e_{\breve{a}}} - \frac{1}{2e_{\breve{a}}} < \frac{k}{e_{\breve{a}}} - r < \frac{k}{e_{\breve{a}}}$$
and
$$\frac{2k}{e_{\breve{a}}} < \frac{k}{e_{\breve{a}}} + \frac{1}{e_{\breve{a}}} - 2r < \frac{2k}{e_{\breve{a}}} + \frac{1}{e_{\breve{a}}},$$
it follows that $\mathbf{U}_{\breve{a}}(\breve{F})_{x_0,m-r} = \mathbf{U}_{\breve{a}}(\breve{F})_{x_0,m}$ and that $\mathbf{U}_{\breve{a}}(\breve{F})_{x_0,m} = \mathbf{U}_{\breve{a}}(\breve{F})_{x_0,m+\frac{1}{e_{\breve{a}}}}$.
- Suppose the residue characteristic of F is 2. Note that the characteristic of F is necessarily 0. So $\omega(\lambda) = -\omega(2) = -e_F$, where e_F is the ramification index of $F/\mathbb{Q}_2$. We have assumed that $0 < r < \frac{1}{2e_{\breve{a}}}$. We have
$$\mathbf{U}_{\breve{a}}(\breve{F})_{x_0,m} = \left\{x_{\breve{a}}(u,v) \in H(L_{\breve{a}}, L_{2\breve{a}}) \mid \omega(u) \geq \frac{k}{e_{\breve{a}}} + \frac{e_F}{2},\ \omega(v-\lambda u\sigma(u)) \geq \frac{2k}{e_{\breve{a}}} + \frac{1}{e_{\breve{a}}}\right\}.$$
. Write $\frac{e_F}{2} = \frac{e_F e_{2\breve{a}}}{e_{\breve{a}}}$. Then we see that
$$\frac{k}{e_{\breve{a}}} + \frac{e_F e_{2\breve{a}}}{e_{\breve{a}}} - \frac{1}{e_{\breve{a}}} < \frac{k}{e_{\breve{a}}} + \frac{e_F e_{2\breve{a}}}{e_{\breve{a}}} - r < \frac{k}{e_{\breve{a}}} + \frac{e_F e_{2\breve{a}}}{e_{\breve{a}}}$$
and
$$\frac{2k}{e_{\breve{a}}} < \frac{2k}{e_{\breve{a}}} + \frac{1}{e_{\breve{a}}} - 2r < \frac{2k}{e_{\breve{a}}} + \frac{1}{e_{\breve{a}}}.$$
It now again follows that $\mathbf{U}_{\breve{a}}(\breve{F})_{x_0,m-r} = \mathbf{U}_{\breve{a}}(\breve{F})_{x_0,m}$ and that $\mathbf{U}_{\breve{a}}(\breve{F})_{x_0,m+r} = \mathbf{U}_{\breve{a}}(\breve{F})_{x_0,m+\frac{1}{e_{\breve{a}}}}$.

(2) Suppose $\alpha \neq 0$. We have assumed $0 < r < \frac{1}{2e_{\breve{a}}}$. Then we have

$$\mathbf{U}_{\breve{a}}(\breve{F})_{x_0,m} = \left\{ x_{\breve{a}}(u,v) \in H(L_{\breve{a}}, L_{2\breve{a}}) \mid \omega(u) \geq \frac{k}{e_{\breve{a}}} - \frac{1}{2}\omega(\lambda),\ \omega(v - \lambda u \sigma(u)) \geq \frac{2k}{e_{\breve{a}}} + \frac{1}{e_a} \right\}.$$

Now, write $\frac{k}{e_{\breve{a}}} - \frac{1}{2}\omega(\lambda) = \frac{1}{2e_{\breve{a}}} + \frac{k'}{e_{\breve{a}}}$. Then, for $0 < r < \frac{1}{2e_{\breve{a}}}$, we have

$$\frac{k'}{e_{\breve{a}}} < \frac{1}{2e_{\breve{a}}} + \frac{k'}{e_{\breve{a}}} - r < \frac{1}{2e_{\breve{a}}} + \frac{k'}{e_{\breve{a}}} < \frac{k'}{e_{\breve{a}}} + \frac{1}{e_{\breve{a}}}$$

and

$$\frac{2k}{e_{\breve{a}}} < \frac{2k}{e_{\breve{a}}} + \frac{1}{e_{\breve{a}}} - r < \frac{2k}{e_{\breve{a}}} + \frac{1}{e_{\breve{a}}}.$$

Now, it is again clear that $\mathbf{U}_{\breve{a}}(\breve{F})_{x_0,m-r} = \mathbf{U}_{\breve{a}}(\breve{F})_{x_0,m}$ and that $\mathbf{U}_{\breve{a}}(\breve{F})_{x_0,m+r} = \mathbf{U}_{\breve{a}}(\breve{F})_{x_0,m+\frac{1}{e_{\breve{a}}}}$.

This finishes the proof of the lemma. □

Next, we recall the definition of the filtration of the root subgroup $\mathbf{U}_a(F)$ for $a \in \Phi$ (cf. [10, §5.1.16 - 5.1.18]). Let $\breve{\Phi}_a := \{\breve{c} \in \breve{\Phi} \mid \breve{c}|_{\mathbf{S}} = a \text{ or } 2a\}$. This is a σ-stable positively closed subset of $\breve{\Phi}$; that is if $\breve{c}_1, \breve{c}_2 \in \breve{\Phi}_a$ are such that $\breve{c} + \breve{c}'$ is a root, then $\breve{c} + \breve{c}' \in \breve{\Phi}_a$. For any fixed ordering, the subset

$$\mathbf{U}_a(\breve{F})_{x_0,r} := \prod_{\breve{c} \in \breve{\Phi}_a, \breve{c}|_{\mathbf{S}} = a} \mathbf{U}_{\breve{c}}(\breve{F})_{x_0,r} \prod_{\breve{c} \in \breve{\Phi}_a^{\mathrm{nd}}, \breve{c}|_{\mathbf{S}} = 2a} \mathbf{U}_{\breve{c}}(\breve{F})_{x_0,2r} \tag{5.2}$$

is a subgroup of $\mathbf{U}_a(\breve{F})$. Let $\mathbf{U}_a(F)_{x_0,r} := \mathbf{U}_a(\breve{F})_{x_0,r} \cap \mathbf{U}_a(F)$. Let Γ'_a be the value set attached to the root a as in [10, Section 5.1.16]. Let $\breve{a} \in \breve{\Phi}_a$ be such that $\breve{a}|_{\mathbf{S}} = a$. Then by [10, Proposition 5.1.19], we have $\Gamma'_a = \Gamma'_{\breve{a}}$. For $a \in \Phi$ and $\breve{a} \in \breve{\Phi}$ such that $\breve{a}|_{\mathbf{S}} = a$, define $e_a := e_{\breve{a}}$. Note that this definition does not depend on the choice of $\breve{a}$ whose restriction to $\mathbf{S}$ is a (see [10, Section 5.1.15]. We have the following lemma.

Lemma 5.6. *Let $a \in \Phi^{\mathrm{red}}$ and let $\breve{a} \in \breve{\Phi}$ such that $\breve{a}|_S = a$. Let $m \in \frac{1}{e_a}\mathbb{Z}$. Let $r \in \mathbb{R}$ be such that $0 < r < \frac{1}{e_a}$ if $2a$ is not a root, and such that $0 < r < \frac{1}{2e_a}$ if $2a$ is a root. Then $\mathbf{U}_a(F)_{x_0,m-r} = \mathbf{U}_a(F)_{x_0,m}$ and $\mathbf{U}_a(F)_{x_0,m+r} = \mathbf{U}_a(F)_{x_0,m+\frac{1}{e_a}}$.*

Proof. Let $m \in \frac{1}{e_a}\mathbb{Z}$ and r as above. We need to show that $\mathbf{U}_a(\breve{F})_{x_0,m-r} = \mathbf{U}_a(\breve{F})_{x_0,m}$. Let $\breve{c} \in \breve{\Phi}_a$ be such that $\breve{c}|_{\mathbf{S}} = a$. Then, since $\Gamma'_{\breve{c}} = \Gamma'_a$ by [10, Proposition 5.1.19] and $e_{\breve{c}} = e_a$, Lemma 5.5 implies that $\mathbf{U}_{\breve{c}}(\breve{F})_{x_0,m-r} = \mathbf{U}_{\breve{c}}(\breve{F})_{x_0,m}$ and that $\mathbf{U}_{\breve{c}}(\breve{F})_{x_0,m+r} = \mathbf{U}_{\breve{c}}(\breve{F})_{x_0,m+\frac{1}{e_a}}$. Next, suppose $\breve{c} \in \breve{\Phi}_a^{\mathrm{nd}}$ is such that $\breve{c}|_{\mathbf{S}} = 2a$. Then there exist distinct $\breve{c}_1, \breve{c}_2 \in \breve{\Phi}$ such that $\breve{c}_1 + \breve{c}_2 = \breve{c}$ and $\breve{c}_1|_{\mathbf{S}} = \breve{c}_2|_{\mathbf{S}} = a$. Further $\Gamma'_{\breve{c}_1} = \Gamma'_{\breve{c}_2} = \Gamma'_{\breve{c}} = \Gamma'_{2a} = \Gamma'_a$ and $e_{\breve{c}_1} = e_{\breve{c}_2} = e_{\breve{c}} = e_{2a} = e_a$. Since we have assumed that $0 < 2r < \frac{1}{e_a}$, we see that

$$2m - \frac{1}{e_a} < 2m - 2r < 2m \quad \text{and} \quad 2m < 2m + 2r < 2m + \frac{1}{e_a}$$

So $\mathbf{U}_{\breve{c}}(\breve{F})_{x_0,2m-2r} = \mathbf{U}_{\breve{c}}(\breve{F})_{x_0,2m}$ and similarly, $\mathbf{U}_{\breve{c}}(\breve{F})_{x_0,2m+2r} = \mathbf{U}_{\breve{c}}(\breve{F})_{x_0,2m+\frac{1}{e_a}}$. This proves that $\mathbf{U}_a(\breve{F})_{x_0,m-r} = \mathbf{U}_a(\breve{F})_{x_0,m}$ and that $\mathbf{U}_a(F)_{x_0,m+r} = \mathbf{U}_a(F)_{x_0,m+\frac{1}{e_a}}$. This finishes the proof of the lemma. □

Let

$$\mathbb{R}_G := \{r \in \mathbb{R}_{\geq 0} \mid r \in \frac{1}{e_a}\mathbb{Z} \text{ for all } a \in \Phi\}. \tag{5.3}$$

Remark 5.7. Note that $\mathbb{N} \subset \mathbb{R}_G$. For example, if $\mathbf{G}$ is a connected, reductive group that splits over an unramified extension of F, we have $\mathbb{R}_G = \mathbb{N}$. If $\mathbf{G} = \operatorname{Res}_{L/F}\mathbf{G}'$, then $\mathbb{R}_G = \frac{1}{e_{L/F}}\mathbb{R}_{G'}$ where $e_{L/F}$ is the ramification index of L/F.

Proposition 5.8. *Let $\mathcal{a}$ be an alcove in $\mathcal{A}(S,F)$. Let $x \in \mathcal{a}$ and let $m \in \mathbb{R}_G$. Then $G_{x,m} \in \mathcal{K}^{\heartsuit}(S,G)$.*

Proof. We only need to verify that $G_{x,m}$ satisfies (1) of Definition 5.1. Let N be the normalizer of S in G. Then, using the Iwasawa decomposition, we know that $gG_{x,m}g^{-1}$ is P-conjugate to $nG_{x,m}n^{-1}$ for a suitable n in N. But $nG_{x,m}n^{-1} = G_{n(x),m}$. So, we only need to verify that $G_{n(x),m} \cap M$ is M-conjugate to $G_{x,m} \cap M$. We may and do assume that $M = M_\theta$ for a suitable $\theta \subset \Delta$. Let Φ_θ be the set of roots in $\Phi(G,S)$ that lie in the $\mathbb{Q}$-span of θ. We accordingly have Φ_θ^+ and Φ_θ^-. Then $W_\theta = \langle s_a \mid a \in \theta\rangle = W(M,S)$. Every element $w \in W(G,S)$ can be written as w_1w_2 where $w_1 \in W_\theta$ and $w_2^{-1}(\theta) > 0$.

To prove (2), it suffices to show that $G_{w_1w_2\cdot x,m} \cap M$ is M-conjugate to $G_{x,m} \cap M$. Since $w_1 \in W_\theta$, we see that $G_{w_1w_2\cdot x,m} \cap M$ is M-conjugate to $G_{w_2\cdot x,m} \cap M$. Hence we only need to show that $G_{w_2\cdot x,m} \cap M = G_{x,m} \cap M$. Since $G_{x,m} = \langle T_m, \mathbf{U}_a(F)_{x,m} \mid a \in \Phi_\theta\rangle$, it suffices show that

$$\mathbf{U}_a(F)_{w_2\cdot x,m} = \mathbf{U}_a(F)_{x,m} \ \ \forall a \in \Phi_\theta. \tag{5.4}$$

But $\mathbf{U}_a(F)_{x,m} = \mathbf{U}_a(F)_{x_0,m-a(x-x_0)}$ and $\mathbf{U}_a(F)_{w_2\cdot x,m} = \mathbf{U}_a(F)_{x_0,m-w_2^{-1}(a)(x-x_0)}$.

Let $s_a = \inf\{s \in \Gamma'_a \mid s > 0\}$. Consider the affine linear functional $\psi_{a,r} : y \to a(y-x_0)+r$ for $a \in \Phi, r \in \mathbb{R}$. This is an affine root precisely when $r \in \Gamma'_a$ by [10, Proposition 4.2.22 and Theorem 5.1.20]. Having chosen $\mathcal{a}$ and x_0, we see that for $a \in \Phi^+$, $\psi_{a,0}$ and ψ_{-a,s_a} are both positive affine roots. Since $x \in \mathcal{a}$, we see that $0 < a(x-x_0) < s_a$.

Note that $\Gamma'_{w_2^{-1}a} = \Gamma'_a$. For $a \in \Phi_\theta^+$, since $w_2^{-1}(a)$ is positive, we have $\psi_{w_2^{-1}(a),0}$ and $\psi_{-w_2^{-1}(a),s_a}$ are also positive affine roots, so $0 < w_2^{-1}(a)(x-x_0) < s_a$.

To prove (5.4), we need to show that for $a \in \Phi_\theta^+$,

$$\mathbf{U}_a(F)_{x_0,m-a(x-x_0)} = \mathbf{U}_a(F)_{x_0,m-w_2^{-1}(a)(x-x_0)} = \mathbf{U}_a(F)_{x_0,m}, \tag{5.5}$$

and that for $a \in \Phi_\theta^-$,

$$\mathbf{U}_a(F)_{x_0,m-a(x-x_0)} = \mathbf{U}_a(F)_{x_0,m-w_2^{-1}(a)(x-x_0)} = \mathbf{U}_a(F)_{x_0,m+\frac{1}{e_a}}. \tag{5.6}$$

We see that (5.5) and (5.6) would follow from Lemma 5.6 as soon as we show that for $a \in \Phi_\theta^+ \cap \Phi_\theta^{\mathrm{red}}$, $a(x-x_0), w_2^{-1}(a)(x-x_0) \in (0, \frac{1}{e_a})$ if $2a$ is not a root and that $a(x-x_0), w_2^{-1}(a)(x-x_0) \in (0, \frac{1}{2e_a})$ if $2a$ is a root. We have a few cases.

- Suppose $a \in \Phi_\theta^{\mathrm{red}} \cap \Phi_\theta^+$ is such that $2a$ is not a root. Then $\Gamma'_a = \frac{1}{e_a}\mathbb{Z}$, so $s_a = \frac{1}{e_a}$. So $a(x-x_0), w_2^{-1}(a)(x-x_0) \in (0, \frac{1}{e_a})$.
- Suppose $a \in \Phi_\theta^{\mathrm{red}} \cap \Phi_\theta^+$ is such that $2a$ is a root and such that $\breve{\Phi}_a^{\mathrm{nd}}$ is non-empty. Then $\Gamma'_a = \Gamma'_{2a} = \frac{1}{e_a}\mathbb{Z}$, so $s_a = s_{2a} = \frac{1}{e_a}$. So $a(x-x_0), w_2^{-1}(a)(x-x_0) \in (0, \frac{1}{e_a})$ and $2a(x-x_0), w_2^{-1}(2a)(x-x_0) \in (0, \frac{1}{e_a})$. In particular, $a(x-x_0), w_2^{-1}(a)(x-x_0) \in (0, \frac{1}{2e_a})$.

- Suppose $a \in \Phi_\theta^{\mathrm{red}} \cap \Phi_\theta^+$ is such that $2a$ is a root and such that $\breve{\Phi}_a^{\mathrm{nd}}$ is empty. Let $\breve{a} \in \breve{\Phi}_a$ be such that $\breve{a}|_{\mathbf{S}} = a$ and $2\breve{a}$ is a root. As recalled in subsection 5.1.1, let $L_{\breve{a}} = L_{2\breve{a}}[t]$ where $t^2 - \alpha t + \beta = 0$.
  - Suppose $\alpha = 0$. By (5.1), $\Gamma'_{\breve{a}} = \frac{1}{e_{\breve{a}}}\mathbb{Z}$. Further, $\Gamma'_{2\breve{a}} = \omega(L^0_{\breve{a}}\backslash\{0\}) = \frac{1}{e_{\breve{a}}} + \frac{2}{e_{\breve{a}}}\mathbb{Z}$, so $s_{\breve{a}} = s_{2\breve{a}} = \frac{1}{e_{\breve{a}}}$. Noting that $e_a = e_{\breve{a}}$, we see that $a(x-x_0), w_2^{-1}(a)(x-x_0) \in (0, \frac{1}{e_a})$ and $2a(x-x_0), w_2^{-1}(2a)(x-x_0) \in (0, \frac{1}{e_a})$. In particular, $a(x-x_0), w_2^{-1}(a)(x-x_0) \in (0, \frac{1}{2e_a})$.
  - Suppose $\alpha \neq 0$. Then $\Gamma'_{\breve{a}} = \frac{1}{2e_{\breve{a}}} + \frac{1}{e_{\breve{a}}}\mathbb{Z}$. So $s_{\breve{a}} = \frac{1}{2e_{\breve{a}}}$. Further, $\Gamma'_{2\breve{a}} = \omega(L^0_{\breve{a}}\backslash\{0\}) = \frac{2}{e_{\breve{a}}}\mathbb{Z}$. Noting that $e_a = e_{\breve{a}}$, we see that $a(x-x_0), w_2^{-1}(a)(x-x_0) \in (0, \frac{1}{2e_a})$.

We have proved (5.4). This finishes the proof of the proposition. □

5.2. **Some compact open subgroups that don't live in $\mathcal{K}^\heartsuit(S,G)$.** It is shown in [3, Section 5], that for each $x \in \mathcal{A}(S,F)$ and each $r > 0$, $G_{x,r} \in \mathcal{K}^\spadesuit(S,G)$. In this subsection, we give examples of $G_{x,r}$'s that do not lie in $\mathcal{K}^\heartsuit(S,G)$. In the proof of Proposition 5.8, we had used crucially that if x is not already a special point, then it lies in the interior of the alcove *and* that $r \in \mathbb{R}_G$ for the argument to go through. This suggests how to look for points $x \in \mathcal{A}(S,F)$ and $r > 0$ for which $G_{x,r} \notin \mathcal{K}^\heartsuit(S,G)$.

5.2.1. *Example.* Let $G = \mathrm{GL}_3$ with the diagonal matrices as T and upper-traingular matrices as B. With this choice, let $\Delta = \{e_1 - e_2, e_2 - e_3\}$. Let $M = \mathrm{GL}_1 \times \mathrm{GL}_2$. Then $\theta = \{e_2 - e_3\}$. Let $a = e_2 - e_3$, let $w = s_{e_1 - e_2}$ and let $x = e_1^*/2$. Then

$$G_{x,1} = \begin{bmatrix} 1+\mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{p}_F \\ \mathfrak{p}_F^2 & 1+\mathfrak{p}_F & \mathfrak{p}_F \\ \mathfrak{p}_F^2 & \mathfrak{p}_F & 1+\mathfrak{p}_F \end{bmatrix}.$$

With

$$n = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

we know that n is a representative of w in GL_3. Now,

$$nG_{x,1}n^{-1} = \begin{bmatrix} 1+\mathfrak{p}_F & \mathfrak{p}_F^2 & \mathfrak{p}_F \\ \mathfrak{p}_F & 1+\mathfrak{p}_F & \mathfrak{p}_F \\ \mathfrak{p}_F & \mathfrak{p}_F^2 & 1+\mathfrak{p}_F \end{bmatrix}.$$

Then

$$G_{x,1} \cap M = \begin{bmatrix} 1+\mathfrak{p}_F & 0 & 0 \\ 0 & 1+\mathfrak{p}_F & \mathfrak{p}_F \\ 0 & \mathfrak{p}_F & 1+\mathfrak{p}_F \end{bmatrix}$$

and

$$nG_{x,1}n^{-1} \cap M = \begin{bmatrix} 1+\mathfrak{p}_F & 0 & 0 \\ 0 & 1+\mathfrak{p}_F & \mathfrak{p}_F \\ 0 & \mathfrak{p}_F^2 & 1+\mathfrak{p}_F \end{bmatrix}.$$

We claim that $nG_{x,1}n^{-1} \cap M$ and $G_{x,1} \cap M$ are **not** M-conjugate. To see this, it suffices to prove that the groups

$$K_1 = \begin{bmatrix} 1+\mathfrak{p}_F & \mathfrak{p}_F \\ \mathfrak{p}_F & 1+\mathfrak{p}_F \end{bmatrix} \text{ and } I_1 = \begin{bmatrix} 1+\mathfrak{p}_F & \mathfrak{p}_F \\ \mathfrak{p}_F^2 & 1+\mathfrak{p}_F \end{bmatrix}$$

are not GL_2-conjugate. This is intuitively clear since $K_1 \subset GL_2(\mathfrak{O}_F)$ which corresponds to the parahoric subgroup of a hyperspecial vertex in the building and $I_1 \subset I = \begin{bmatrix} \mathfrak{O}_F^\times & \mathfrak{O}_F \\ \mathfrak{p}_F & \mathfrak{O}_F^\times \end{bmatrix}$ which is an Iwahori subgroup of $\mathrm{GL}_2(F)$ and corresponds to the parahoric subgroup of the interior of an alcove. We justify this as follows. Normalize the Haar measure on $\mathrm{GL}_2(F)$ so that $\mathrm{vol}(I) = 1$. Note that K_1 and I_1 are both normal subgroups of I and $I_1 \subsetneq K_1 \subsetneq I$. Now, if K_1 and I_1 are GL_2-conjugate, then $\mathrm{vol}(K_1) = \mathrm{vol}(I_1)$, which then implies that $[I : K_1] = \mathrm{vol}(K_1)^{-1} = \mathrm{vol}(I_1)^{-1} = [I : I_1]$, which is not possible. This proves that with $x = e_1^*/2$, $G_{x,1} \notin \mathcal{K}^\heartsuit(S, G)$.

5.2.2. *Example.* Let $G = \mathrm{GL}_4$ with the diagonal matrices as T and upper-traingular matrices as B. With this choice, let $a_i = e_i - e_{i+1}$, $1 \le i \le 3$. Then $\Delta = \{a_1, a_2, a_3\}$. Let $M = \mathrm{GL}_2 \times \mathrm{GL}_2$. Then $\theta = \{a_1, a_3\}$. Let $w = s_{a_2}$ and let $x = \dfrac{3a_1^\vee + 4a_2^\vee + 3a_3^\vee}{8}$. Then x is the barycenter of the alcove $\mathcal{a}$ whose bounding hyperplanes are given by the affine roots $a_1, a_2, a_3, 1 - (a_1 + a_2 + a_3)$. Let $r = 3/8$. Then

$$G_{x,3/8} = \begin{bmatrix} 1+\mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{O}_F & \mathfrak{O}_F \\ \mathfrak{p}_F & 1+\mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{O}_F \\ \mathfrak{p}_F & \mathfrak{p}_F & 1+\mathfrak{p}_F & \mathfrak{p}_F \\ \mathfrak{p}_F^2 & \mathfrak{p}_F & \mathfrak{p}_F & 1+\mathfrak{p}_F \end{bmatrix}.$$

With

$$n = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

we know that n is a representative of w in GL_4. Now,

$$nG_{x,3/8}n^{-1} = \begin{bmatrix} 1+\mathfrak{p}_F & \mathfrak{O}_F & \mathfrak{p}_F & \mathfrak{O}_F \\ \mathfrak{p}_F & 1+\mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{p}_F \\ \mathfrak{p}_F & \mathfrak{p}_F & 1+\mathfrak{p}_F & \mathfrak{O}_F \\ \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{p}_F & 1+\mathfrak{p}_F \end{bmatrix}.$$

Then

$$G_{x,3/8} \cap M = \begin{bmatrix} 1+\mathfrak{p}_F & \mathfrak{p}_F & 0 & 0 \\ \mathfrak{p}_F & 1+\mathfrak{p}_F & 0 & 0 \\ 0 & 0 & 1+\mathfrak{p}_F & \mathfrak{p}_F \\ 0 & 0 & \mathfrak{p}_F & 1+\mathfrak{p}_F \end{bmatrix}.$$

and

$$nG_{x,3/8}n^{-1} \cap M = \begin{bmatrix} 1+\mathfrak{p}_F & \mathfrak{O}_F & 0 & 0 \\ \mathfrak{p}_F & 1+\mathfrak{p}_F & 0 & 0 \\ 0 & 0 & 1+\mathfrak{p}_F & \mathfrak{O}_F \\ 0 & 0 & \mathfrak{p}_F & 1+\mathfrak{p}_F \end{bmatrix}.$$

Clearly, $G_{x,3/8} \cap M$ and $nG_{x,3/8}n^{-1} \cap M$ are not M-conjugate.

6. A special case

Let $\mathbf{G}$ be a connected, reductive group over F, $\mathbf{S}$ a maximal F-split torus in $\mathbf{G}$, $\mathcal{A}(S,F)$ an apartment of $\mathbf{S}$ over F. Let I be the Iwahori subgroup corresponding to an alcove in $\mathcal{A}(S,F)$. Let $\mathbf{M}_0$ be a minimal Levi subgroup of $\mathbf{G}$ that contains $\mathbf{S}$ and $\mathbf{T}$ a maximal torus in $\mathbf{M}_0$ that contains $\mathbf{S}$. We know that the center of the Iwahori Hecke algebra $\mathcal{H}(G,I)$ is given by $\mathbb{C}[\Omega_{M_0}]^{W_0}$ where Ω_{M_0} is the image of the Kottwitz homomorphism κ_{M_0} (see [15, Section 7]) and W_0 is the Weyl group of S in G. When G is split, we have $M_0 = S = T$ and the center is given by $\mathbb{C}[X_{\mathrm{nr}}(T)]^{W_0}$ or in other words is $\mathcal{H}(T,T_0)^{W_0}$. However, for Hecke algebras at deeper level, one cannot hope for such a simple description of the center. In fact, we will now show that $\mathcal{H}(T,T_n)^{W_0}$ captures *only a part of the center* of only the principal series blocks. We illustrate this when G is split.

Assume now that $\mathbf{G}$ is a split connected, reductive group over $\mathbb{Z}$. In this case, $\mathbf{M} = \mathbf{T}$ is a maximal split torus in $\mathbf{G}$. Let $\Phi = \Phi(\mathbf{G},\mathbf{T})$ be the set of roots of T in G, $\Phi^\vee$ the set of coroots and let $W_0 = W(\mathbf{G},\mathbf{T})$ denote the absolute Weyl group. Let $\mathbf{B}$ be a Borel subgroup of G containing $\mathbf{T}$. This determines a set of simple roots Δ of Φ. Let $X_{\mathrm{nr}}(T)_+ = \{\lambda \in X_{\mathrm{nr}}(T) \mid \langle \alpha, \lambda\rangle \geq 0 \ \forall \alpha \in \Delta\}$. Let W denote the Iwahori Weyl group of G, I be an Iwahori subgroup of G and let I_n be the n-th Moy–Prasad filtration subgroup of I. Let $T_n = T \cap I_n$. The goal of this section is to prove the following proposition. All the crucial ingredients needed for this proposition are contained in the work of Roche [20].

Proposition 6.1. *Let $p = char(k_F)$ be large enough satisfying the hypothesis for Theorem 4.15 of* [20]*. There exists an algebra embedding $t_{B,n} : \mathcal{H}(T,T_n) \to \mathcal{H}(G,I_n)$ with the following properties:*

(1) $t_{B,n}(\mathcal{H}(T,T_0)) \subset \mathcal{H}(G,I_0)$.

(2) The sub-algebra $t_{B,n}(\mathcal{H}(T,T_n)^{W_0}) \subset \mathfrak{Z}(\mathcal{H}(G,I_n))$. In fact

$$t_{B,n}(\mathcal{H}(T,T_n)^{W_0}) \subset \prod_{\chi \in (T_0/T_n)^\vee} t_{B,\chi}(\mathcal{H}(T,\chi)^{W_\chi}) \simeq \prod_{\chi \in (T_0/T_n)^\vee} \mathfrak{Z}(\mathfrak{R}_\chi(G))$$

where $\mathfrak{R}_\chi(G)$ is the principal series block determined by the inertial equivalence class of the character χ of T_0/T_n, and $W_\chi = \{w \in W_0 \mid w \cdot \chi = \chi\}$.

First, we present a direct proof that incorporates some of the previously established results. Subsequently, we give a more explicit proof which uses a theorem of Roche.

Proof. The two pairs $(I_n, 1_{I_n})$ and $(T_n, 1_{T_n})$ are types in G and in T, respectively. Moreover, $(T_0, 1_{T_n}) \in \mathcal{J}_T^{wc}$. It is immediate to show (or apply Theorem 3.2)

$$\mathfrak{S}_{T_n} = \{\mathfrak{t}_\chi = [{}^0T, \bar{\chi}]_T : \chi \in X({}^0T/T_n)\},$$

where, $\bar{\chi}$ denotes any extension of χ to T. By Lemma 5.3 we know that

$$\{\mathfrak{s}_\chi = [{}^0T, \chi]_G : \chi \in X({}^0T/T_n)\} \subset \mathfrak{S}_{I_n}.$$

Note that for any $\chi \in X({}^0T/T_n)$, we have $\bar{\chi}^n \simeq \nu \otimes \bar{\chi}$ for some $\nu \in X(T/{}^0T)$ if and only if $\chi^n = \chi$. Then

$$N^{\mathfrak{t}} = N^\chi = \{n \in N_G(T) : \chi^n = \chi\}, \text{ So } W^{\mathfrak{t}} = W_\chi = \{w \in W_0 : \chi^w = \chi\}.$$

By Bernstein decomposition (§1.1) (or apply Corollary 4.4) we have

$$\mathcal{H}(T,T_N)=\mathcal{Z}(\mathfrak{R}_{T_n}(T))=\bigoplus_{\chi\in X({}^0T/T_n)}\mathfrak{z}^{\mathfrak{t}_\chi}=\bigoplus_{\chi\in X({}^0T/T_n)}\mathcal{H}(T,\chi).$$

Note that W_0 normalizes T_n, so it acts by conjugation on $X({}^0T/T_n)$ and also on $\mathcal{H}(T,T_N)$. Accordingly,

$$\mathcal{H}(T,T_N)^{W_0}=\left(\bigoplus_{\chi\in X({}^0T/T_n)}\mathfrak{z}^{\mathfrak{t}_\chi}\right)^{W_0}=\bigoplus_{[\chi]\in X({}^0T/T_n)/W_0}(\mathfrak{z}^{\mathfrak{t}_\chi})^{W_\chi}\subset\bigoplus_{\chi\in X({}^0T/T_n)}(\mathfrak{z}^{\mathfrak{t}_\chi})^{W_\chi}.$$

where $[\chi]$ denotes the W_0-orbit of χ in $X({}^0T/T_n)$. Using Theorem 4.8 we conclude

$$\bigoplus_{\chi\in X({}^0T/T_n)}(\mathfrak{z}^{\mathfrak{t}_\chi})^{W_\chi}=\bigoplus_{\chi\in X({}^0T/T_n)}\mathfrak{z}^{\mathfrak{s}_\chi}\subset\mathcal{Z}(\mathfrak{R}_{I_n}(G))=\mathcal{H}(G,I_n).\qquad\square$$

In fact, the assumption "G is split" was not used in the previous proof. Proposition 5.8 guarantees that the same outcome holds for any general reductive connected group, with $T=M_0$ as a minimal Levi. However, if G is split, a more explicit construction of the above isomorphism is possible using[20]:

Proof. The idea is to decompose $\mathcal{H}(T,T_n)$ and $\mathcal{H}(G,I_n)$ in accordance with the Bernstein decomposition and construct the required embedding block wise. First, note that there is a $\mathbb{C}$-algebra isomorphism

$$\mathcal{H}(T,T_n)\simeq\prod_{\chi\in(T_0/T_n)^\vee}\mathcal{H}(T,\chi).\tag{6.1}$$

In fact, we have an exact sequence $1\to T_0/T_n\to T/T_n\to X_{\mathrm{nr}}(T)\to 1$. Fixing a uniformizer ϖ_F of F yields the splitting $\mu\mapsto\mu(\varpi_F)\bmod T_n$, hence

$$T/T_n\simeq X_{\mathrm{nr}}(T)\times T_0/T_n.$$

Accordingly, $\mathcal{H}(T,T_n)\simeq\mathbb{C}[X_{\mathrm{nr}}(T)]\otimes_{\mathbb{C}}\mathbb{C}[T_0/T_n]$. Now decomposing the $\mathbb{C}$-algebra $\mathbb{C}[T_0/T_n]$ using Artin–Wedderburn theorem proves (6.1).

For each $\chi\in X({}^0T/T_n)$, Roche has constructed in [20, Section 4 and Theorem 7.7]) a G-cover for (T,χ) denoted (J_χ,ρ_χ). It is clear from the construction above that $I_n\subset J_\chi$ and $\rho_\chi|_{I_n}=1$. In particular, $\mathcal{H}(G,\rho_\chi)\subset\mathcal{H}(G,I_n)$.

For each $\chi\in X({}^0T/T_n)$, Roche constructs[3] in [20, Section 5] an algebra isomorphism $t_{B,\chi}:\mathcal{H}(T,\chi)\to\mathcal{H}(G,\rho_\chi)$ which preserve canonical involution's and inner products. In more words, the choice of ϖ_F splits the exact sequence $1\to T_0\to T(F)\to X_{\mathrm{nr}}(T)\to 1$. Let $\phi:X_{\mathrm{nr}}(T)\to T(F),\lambda\to\lambda(\varpi_F)$ be the homomorphism that splits the sequence above. For $\lambda\in X_{\mathrm{nr}}(T)$, let $f_{\lambda,\chi}\in\mathcal{H}(T,\chi)$ be the unique function with support in $T_0\lambda$ and such that $f_{\lambda,\chi}(\lambda(\varpi_F))=1$. The functions $f_{\lambda,\chi},\lambda\in X_{\mathrm{nr}}(T)$, forms a $\mathbb{C}$-basis of $\mathcal{H}(T,\chi)$. For each $\lambda\in X_{\mathrm{nr}}(T)_+$, let $\widetilde{f}_{\lambda,\chi}$ be the unique element of $\mathcal{H}(G,\rho_\chi)$ with support in $J_\chi\lambda J_\chi$ such that $\widetilde{f}_{\lambda,\chi}(\lambda(\varpi_F))=1$. This is invertible in $\mathcal{H}(G,\rho_\chi)$ by [20, Lemma 7.6]. For $\lambda\in X_{\mathrm{nr}}(T)$, write $\lambda=\lambda_1-\lambda_2$ with $\lambda_1,\lambda_2\in X_{\mathrm{nr}}(T)_+$. Let $t_{B,\chi}(f_{\lambda,\chi})=\widetilde{f}_{\lambda_1,\chi}\widetilde{f}^{-1}_{\lambda_2,\chi}$. This is a $\mathbb{C}$-algebra embedding of $\mathcal{H}(T,\chi)$ into $\mathcal{H}(G,\rho_\chi)$ (see [20, Section 9]).

Using (6.1), we get the required embedding $t_{B,n}:\mathcal{H}(T,T_n)\to\mathcal{H}(G,I_n)$.

Note that $\mathcal{H}(T,T_0)=\mathcal{H}(T,\chi_0)$ where χ_0 is the trivial character of T_0. Then $J_{\chi_0}=I$, ρ_{χ_0} is the trivial character of J_{χ_0} and $\mathcal{H}(G,\rho_{\chi_0})=\mathcal{H}(G,I)$ is the Iwahori

[3]Roche's construction is derived from Bushnell and Kutzko isomorphism [5, Section 7] by twisting by the square-root of the modulus character of B)

Hecke algebra. Further, $t_{B,n}|_{\mathcal{H}(T,T_0)} = t_{B,\chi_0}$ has the property stated in (1) by construction.

Let us prove (2). Recall that $\mathfrak{R}_\chi(G)$ is equivalent to $\mathcal{H}(G,\rho_\chi)-$mod and $\mathfrak{R}_\chi(T)$ is equivalent to $\mathcal{H}(T,\chi)-$mod. Let $\tilde{\chi}$ be any extension of χ to T and let $\mathfrak{t} = [T,\tilde{\chi}]_T \in \mathfrak{B}(T)$ be the inertial equivalence class of $\mathfrak{R}_\chi(T)$. Then

$$\mathfrak{Z}(\mathfrak{R}_\chi(G)) = (t_{B,\chi}(\mathcal{H}(T,\chi)^{W_\chi}).$$

It remains to see that

$$t_{B,n}(\mathcal{H}(T,T_n)^{W_0}) \subset \prod_{\chi\in(T_0/T_n)^\vee} t_{B,\chi}(\mathcal{H}(T,\chi)^{W_\chi}).$$

The elements $\{f_{\lambda,\chi} \mid \lambda \in X_{\mathrm{nr}}(T), \chi \in (T_0/T_n)^\vee\}$ forms a basis for $\mathcal{H}(T,T_n)$. The elements $z_{\mathcal{O}} = \sum_{(\lambda,\chi)\in\mathcal{O}} f_{\lambda,\chi}$ for all W_0-orbits $\mathcal{O}$ in $X_{\mathrm{nr}}(T) \times (T_0/T_n)^\vee$ then forms a basis for $\mathcal{H}(T,T_n)^{W_0}$. So it suffices to prove that for each such W_0-orbit $\mathcal{O}$, $z_{\mathcal{O}} \in \prod_{\chi\in(T_0/T_n)^\vee} \mathcal{H}(T,\chi)^{W_\chi}$.

Write $\mathcal{O} = \{(w(\lambda_\dagger), w(\chi_\dagger)) \mid w \in W_0\}$ for a fixed $\lambda_\dagger$ and $\chi_\dagger$. Given $\chi = w_0(\chi_\dagger)$ in the W_0-orbit of $\chi_\dagger$, set $\lambda = w_0(\lambda_\dagger)$ and let $\mathcal{O}_\chi = \{(w(\lambda),\chi) \mid w \in W_\chi\}$. Let $z_{\mathcal{O}_\chi} = \sum_{(\lambda',\chi')\in\mathcal{O}_\chi} f_{\lambda',\chi'}$. Note that $z_{\mathcal{O}_\chi} \in \mathcal{H}(T,\chi)^{W_\chi}$. For each $w \in W_0$, it is easy to see that $w\cdot\mathcal{O}_\chi = \mathcal{O}_{(w\cdot\chi)}$. It is clear that

$$\mathcal{O} = \bigsqcup_{\chi\in W_0-\text{orbit of } \chi_\dagger} \mathcal{O}_\chi.$$

Then

$$z_{\mathcal{O}} = \sum_{\chi\in W_0-\text{orbit of } \chi_\dagger} z_{\mathcal{O}_\chi}.$$

Since $z_{\mathcal{O}_\chi} \in \mathcal{H}(T,\chi)^{W_\chi}$, it follows that $z_{\mathcal{O}} \in \prod_{\chi\in(T_0/T_n)^\vee} \mathcal{H}(T,\chi)^{W_\chi}$. This finishes the proof of the proposition. □

Remark 6.2. Normalize the Haar measure on G so that the volume of I_0 is 1. For $\lambda \in X_{\mathrm{nr}}(T)$, let $q(\lambda) = [I_0\lambda I_0 : I_0]$ and define $\mathcal{T}_\lambda = q(\lambda)^{-1}1_{I_0\lambda I_0} \in \mathcal{H}(G,I_0)$. Any λ in $X_{\mathrm{nr}}(T)$ can be written as $\lambda_1 - \lambda_2$ for $\lambda_i \in X_{\mathrm{nr}}(T)_+$. For $\lambda \in X_{\mathrm{nr}}(T)_+$, $\mathcal{T}_\lambda$ is invertible. Set $\Theta_\lambda = \mathcal{T}_{\lambda_1}\mathcal{T}_{\lambda_2}^{-1}$. Let χ_0 be the trivial character of T_0. Identify $\mathcal{H}(T,\chi_0)$ with $\mathbb{C}[X_{\mathrm{nr}}(T)]$. In this situation, $t_{B,n} : \mathbb{C}[X_{\mathrm{nr}}(T)] \to \mathcal{H}(G,I_0)$ is given by $t_{B,n}(\lambda) = t_{B,\chi_0}(\lambda) = \Theta_\lambda$ by construction, so $t_{B,n}$ restricted to $\mathbb{C}[X_{\mathrm{nr}}(T)]$ agrees with the Bernstein-Lusztig embedding for the Iwahori Hecke algebra (see [22, Section 5] or [4, Section 4.3]).

7. The Bernstein center at deeper level

Let F be a non-archimedean local field and let $\mathbf{G}$ be a connected, reductive group over F. We assume the following.

Assumption 7.1. $\mathbf{G}$ splits over a tamely ramified extension of F, and the residue characteristic p of F does not divide the order of the Weyl group of $\mathbf{G}$.

(1) By [12, Theorem 8.1] Every irreducible, supercuspidal representation of G arises from Yu's construction, that was recalled in Section 7.1.
(2) Let (π,V) be an irreducible, smooth representation of G and let $\mathfrak{s} = [M,\sigma]_G$ be the inertial class of π. Let (J_M,ρ_M) be a supercuspidal type of the Bernstein block corresponding to $\mathfrak{s}_M = [M,\sigma]_M$. Then there exists a G-cover (J,ρ) of (J_M,ρ_M), which in particular says that (J,ρ) is an $\mathfrak{s}$-type. This construction

is carried out in [14, Theorem 9.1] under some additional hypothesis, but in [12, Theorem 7.12], the author has proved that this holds merely under Assumption 7.1.

For the remainder of this section we assume that Assumption 7.1 holds. We now recall Yu's construction of supercuspidal representations.

7.1. **Yu's construction of supercuspidal representations and a corollary.** The Yu datum consists of a 5-tuple $(\vec{\mathbf{G}}, y, \vec{r}, \rho_{-1}, \vec{\phi})$ where

D1: $\vec{\mathbf{G}} = (\mathbf{G}^0, \cdots, \mathbf{G}^d)$ is a tower of algebraic subgroups of $\mathbf{G}$,

$$\mathbf{G}^0 \subsetneq \cdots \subsetneq \mathbf{G}^d = \mathbf{G}$$

such that $Z(\mathbf{G}^0)/Z(\mathbf{G})$ is anisotropic over F and $\vec{\mathbf{G}}$ is a tamely ramified twisted Levi sequence in $\mathbf{G}$ in the section of [25, Section 1]. In particular, $\mathbf{G}^i \otimes F^t$ is split and is a Levi factor of a parabolic subgroup of $\mathbf{G} \otimes F^t$.

D2: y is a point in $\mathcal{B}(\mathbf{G}, F) \cap \mathcal{A}(\mathbf{G}, T, E)$ where $\mathbf{T}$ is a maximal torus in $\mathbf{G}^0$, E is a Galois tamely ramified splitting extension of T and hence of $\vec{\mathbf{G}}$.

D3: $\vec{r} = (r_0, \cdots, r_d)$ is a sequence of real numbers satisfying $0 < r_0 \cdots < r_{d-1} \leq r_d$ if $d > 0$ and $0 \leq r_0$ if $d = 0$.

D4: ρ_{-1} is an irreducible representation of $K^0 = G^0_{[y]}$ such that $\rho_{-1}|_{G^0_{y,0+}}$ is 1-isotypic and the compactly induced representation $\pi_{-1} = \mathrm{ind}_{K^0}^{G^0} \rho_{-1}$ is irreducible, supercuspidal. Here $[y]$ is the projection of y on the reduced building and $G^0_{[y]}$ is the subgroup of G fixing $[y]$.

D5: $\vec{\phi} = (\phi_0, \cdots, \phi_d)$ is such that each ϕ_i is a quasi-character of G^i for each i. We assume that ϕ_i is trivial on $G^i_{r_i,+}$ but not on $G^i_{r_i}$ that is, that $depth(\phi_i) = r_i$ for $0 \leq i \leq d-1$. Here that we have used the convention $G^i_{r_i} = G^i_{y,r_i}$ and similarly for $G^i_{r_i+}$. If $r_{d-1} < r_d$ we assume that ϕ_d is trivial on $G^d_{r_d,+}$ but not on $G^d_{r_d}$, otherwise assume that $\phi_d = 1$. We also assume that ϕ_i is G^{i+1} generic (see [25, Section 9].

Starting with such a datum, Yu's construction gives a supercuspidal representation of depth r_d. Let us summarize this construction. Let $K^0_+ = G^0_{0+}$ and for $1 \leq i \leq d$, let $s_i = r_i/2$ and let

$$K^i = K^0 G^1_{s_0} \cdots G^i_{s_{i-1}}$$

and

$$K^i_+ = K^0_+ G^1_{s_0+} \cdots G^i_{s_{i-1}+}$$

Yu also defines subgroups $J^i, J^i_+, 1 \leq i \leq d$ as follows. Let

$$J^i = \mathbf{G}(F) \cap \langle \mathbf{U}_\alpha(E)_{y,r_{i-1}}, \mathbf{U}_\beta(E)_{y,r_{i-1}/2} \mid \alpha \in \Phi(\mathbf{G}^{i-1}, T, E) \cup \{0\}, \beta \in \Phi(\mathbf{G}^i, T, E) \backslash \Phi(\mathbf{G}^{i-1}, T, E) \rangle$$

and

$$J^i_+ = \mathbf{G}(F) \cap \langle \mathbf{U}_\alpha(E)_{y,r_{i-1}} \mathbf{U}_\beta(E)_{y,r_{i-1}/2+} \mid \alpha \in \Phi(\mathbf{G}^{i-1}, T, E) \cup \{0\}, \beta \in \Phi(\mathbf{G}^i, T, E) \backslash \Phi(\mathbf{G}^{i-1}, T, E) \rangle.$$

For $1 \leq i \leq d$, we have

$$K^{i-1} J^i = K^i, \quad K^{i-1}_+ J^i_+ = K^i_+.$$

Yu's construction of the supercuspidal representation of G from this data is done inductively and includes the following steps.

(a) In [25, Section 11], for $0 \le i \le d-1$, Yu constructs an irreducible representation $\tilde{\phi}_{i-1}$ of $K^{i-1} \ltimes J^i$ using the character ϕ_{i-1} of G^{i-1}, that satisfies condition $\mathbf{SC2}_i$ in [25, Section 4, Page 592]. Let us recall this construction. Let $\hat{\phi}_{i-1}$ be the character of $K^0 G_0^{i-1}\mathbf{G}(F)_{y,s_{i-1}+}$ as in [25, Section 4, Page 591]. Using $\hat{\phi}_{i-1}$, he defines a non-degenrate $\mathbb{F}_p$-valued pairing on the $\mathbb{F}_p$-vector space J^i/J^i_+ making J^i/J^i_+ a symplectic space over $\mathbb{F}_p$. Let $(J^i/J^i_+)^\#$ be the Heisenberg group of J^i/J^i_+. Yu constructs a canonical isomorphism

$$j : J^i/(J^i_+ \cap \ker(\hat{\phi}_{i-1}) \to (J^i/J^i_+)^\#$$

in [25, Proposition 11.4]. Note that K^i act on J^i/J^i_+ by conjugation and this gives a homomorphism from $K^i \to \mathrm{Sp}(J^i/J^i_+)$. Let $\tilde{\phi}_{i-1}$ be the pull back of the Weil representation of $\mathrm{Sp}(J^i/J^i_+) \ltimes (J^i/J^i_+)^\#$ via the map $K^{i-1} \ltimes J^i \to \mathrm{Sp}(J^i/J^i_+) \ltimes (J^i/J^i_+)^\#$. He shows in [25, Theorem 11.5] that $\tilde{\phi}_{i-1}|_{J^i_+}$ is $\hat{\phi}_{i-1}|_{J^i_+}$-isotypic and that the restriction of $\tilde{\phi}_{i-1}$ to K^{i+1}_+ is 1-isotypic.

(b) Next, Yu constructs a representation ρ'_i of K^i such that $\rho'_i|_{G^i_{r_i}}$ is 1-isotypic. He then sets $\rho_i = \rho'_i \otimes \phi_i|_{K^i}$. First, put $\rho'_0 = \rho_{-1}$ and $\rho_0 = \rho'_0 \otimes (\phi_0|_{K^0})$. Now suppose that ρ'_{i-1} and ρ_{i-1} have already been constructed. Inflate $\phi_{i-1}|_{K^{i-1}}$ to a representation $\inf(\phi_{i-1})$ of $K^{i-1} \ltimes J^i$. He shows that the representation $\inf(\phi_{i-1}) \otimes \tilde{\phi}_{i-1}$ factors through the natural map $K^{i-1} \ltimes J^i \to K^{i-1}J^i = K^i$. Let ϕ'_{i-1} be the representation of K^i whose inflation to $K^{i-1} \ltimes J^i$ is $\inf(\phi_{i-1}) \otimes \tilde{\phi}_{i-1}$. Inflate ρ'_{i-1} to a representation $\inf(\rho'_{i-1})$ of $K^i = K^{i-1}J^i$ via the map $K^i \to K^{i-1}J^i/J^i = K^{i-1}/K^{i-1} \cap J^i$ (This can be done because ρ'_{i-1} restricted to $K^{i-1} \cap J^i$ is 1-isotypic). Set $\rho'_i = \inf(\rho'_{i-1}) \otimes \phi'_{i-1}$ and $\rho_i = \rho'_i \otimes (\phi_i|_{K^i})$.

(c) The main theorem of Yu's paper [25] says that the compactly induced representation $\pi_i = \mathrm{ind}_{K^i}^{G^i} \rho_i$ of G^i is irreducible and supercuspidal of depth r_i, $0 \le i \le d$. We note that the proof of this main theorem in Yu's paper relied on some false propositions in literature. Recently, Fintzen gave an alternate proof of the main theorem in [13].

Let ${}^0K^0 = G^0_y$ and ${}^0K^i = ({}^0K^0)G^1_{s_0} \cdots G^i_{s_{i-1}}$. Let ${}^0\rho_i$ be an irreducible summand of $\rho_i|_{{}^0K^i}$. As noted in [25, Corollary 15.3], we have that $({}^0K^i, {}^0\rho_i)$ is a $[G^i, \pi_i]_{G^i}$-type for $0 \le i \le d$.

Lemma 7.2. *For each $0 \le i \le d$,*

(1) Let ${}^0\rho_i$ be any irreducible summand of $\rho_i|_{{}^0K^i}$. Then every $g \in G^i$ that intertwines ${}^0\rho_i$ lies in K^i, and

(2) $m_{{}^0K^0}(\rho_{-1}) = m_{{}^0K^i}(\rho_i)$.

Proof. (1) is a consequence of [25, Corollary 15.5], whose corrected proof can be found in [19, Proposition 4.4]. In more detail, it is shown in loc. cit that if $g \in G^i$ intertwines ${}^0\rho_i$ then $g \in ({}^0K^i)G^0({}^0K^i)$ and that $g \in G^0$ intertwines ${}^0\rho_i$, then $g \in K^0$. Hence $g \in G^i$ intertwines ${}^0\rho_i$, then $g \in ({}^0K^i)(K^0)({}^0K^i)$. By definition $(K^0)({}^0K^i) = K^i$ and clearly $({}^0K^i)(K^i) = K^i$, hence (1) follows.

Let us prove (2). Recall that for $0 \le i \le d$, $\rho_i = \rho'_i \otimes (\phi_i|_{K^i})$, so $\rho_i|_{{}^0K^i} = \rho'_i|_{{}^0K^i} \otimes (\phi_i|_{{}^0K^i})$. Hence $m_{{}^0K^i}(\rho_i) = m_{{}^0K^i}(\rho'_i)$. To prove (2), it suffices to show that $m_{{}^0K^i}(\rho'_i) = m_{{}^0K^{i-1}}(\rho'_{i-1})$. Note that ${}^0K^i ={}^0 K^{i-1}J^i$, and hence under the map

$$K^i \to K^{i-1}J^i/J^i = K^{i-1}/K^{i-1} \cap J^i, \tag{7.1}$$

we have

$$ {}^0K^i \to {}^0 K^{i-1}J^i/J^i = {}^0 K^{i-1}/{}^0K^{i-1} \cap J^i. \tag{7.2}$$

Recall that $\inf(\rho'_{i-1})$ is the inflation of ρ'_{i-1} to $K^i = K^{i-1}J^i$ via (7.1). So the inflation of $\rho'_{i-1}|_{{}^0K^{i-1}}$ to a representation of ${}^0K^i$ via (7.2), denoted $\inf(\rho'_{i-1}|_{{}^0K^{i-1}})$, is precisely $\inf(\rho'_{i-1})|_{{}^0K^i}$ as representations of ${}^0K^i$. In particular,

$$\rho'_i|_{{}^0K^i} = \inf(\rho'_{i-1})|_{{}^0K^i} \otimes (\phi'_{i-1}|_{{}^0K^i}) = \inf(\rho'_{i-1}|_{{}^0K^{i-1}}) \otimes (\phi'_{i-1}|_{{}^0K^i}).$$

Next, we observe that $(\phi'_{i-1}|_{{}^0K^i})$ is irreducible. This is in fact clear from Yu's construction of $\tilde{\phi}_{i-1}$; in more detail, Note that ${}^0K^i$ also acts on J^i/J^i_+ by conjugation and this gives a homomorphism from ${}^0K^i \to \mathrm{Sp}(J^i/J^i_+)$. Let ${}^0\tilde{\phi}_{i-1}$ be the pull back of the Weil representation of $\mathrm{Sp}(J^i/J^i_+) \ltimes (J^i/J^i_+)^\#$ via the map ${}^0K^{i-1} \rtimes J^i \to \mathrm{Sp}(J^i/J^i_+) \rtimes (J^i/J^i_+)^\#$. Then clearly, $(\tilde{\phi}_{i-1}|_{{}^0K^i}) = {}^0\tilde{\phi}_{i-1}$ is irreducible. This proves that $\phi'_{i-1}|_{{}^0K^i} = \inf(\phi_{i-1})|_{{}^0K^i} \otimes (\tilde{\phi}_{i-1}|_{{}^0K^i}) = \inf(\phi_{i-1})|_{{}^0K^i} \otimes ({}^0\tilde{\phi}_{i-1})$ is irreducible.

The construction of $\tilde{\phi}_{i-1}$ recalled above also shows that $\phi'_{i-1}|_{J^i}$ is irreducible. Now, let ${}^0\rho'_0$ be an irreducible summand of $\rho'_0|_{{}^0K^0}$ and set ${}^0\rho'_i := \inf({}^0\rho'_{i-1}) \otimes (\phi'_{i-1}|_{{}^0K^i})$ where $\inf({}^0\rho'_{i-1})$ is the representation of ${}^0K^i$ via (7.2). Note that

$$\mathrm{End}_{\mathbb{C}[{}^0K^i]}({}^0\rho'_i) = \mathrm{End}_{\mathbb{C}[{}^0K^{i-1}]}({}^0\rho'_{i-1})$$

because J^i acts trivially on $\inf({}^0\rho'_{i-1})$ and because $\phi'_{i-1}|_{J^i}$ is irreducible. Hence ${}^0\rho'_i$ is an irreducible summand of $\rho'_i|_{{}^0K^i}$. To finish the proof of (2), we need to show that the

$$\dim_{\mathbb{C}} \mathrm{Hom}_{{}^0K^i}({}^0\rho'_i, \rho'_i) = \dim_{\mathbb{C}} \mathrm{Hom}_{{}^0K^{i-1}}({}^0\rho'_{i-1}, \rho'_{i-1}).$$

This again holds because J^i acts trivially on $\inf({}^0\rho'_{i-1})$ and on $\inf(\rho'_{i-1})$ by (7.1) and (7.2) respectively, and because $\phi'_{i-1}|_{J^i}$ is irreducible. □

Corollary 7.3. *Let G be a connected, reductive group over F and let $\pi_{-1} = \mathrm{ind}_{K^0}^{G^0}\rho_{-1}$ be the depth zero supercuspidal representation of G^0 that is part of the Yu datum. Let $\pi = \pi_d$ be a tame supercuspidal representation of G of depth r_d, obtained as in Yu's construction above. Let $({}^0K^d, {}^0\rho_d)$ be the corresponding $[G, \pi_d]_G$-type described above. Then*

(1) $\pi_{-1}|_{{}^0G^0}$ is multiplicity free if and only if $\pi_d|_{{}^0G^d}$ is multiplicity free.

(2) $\mathcal{H}(G^0, {}^0\rho_{-1})$ is commutative if and only if $\mathcal{H}(G, {}^0\rho_d)$ is commutative.

Proof. This corollary follows from Lemma 7.2 and Proposition 3.14. □

7.2. **The Bernstein center of $\mathcal{H}(G,K)$.** We now assimilate the results in the preceding sections to describe the Bernstein center of $\mathcal{H}(G,K)$ for $K \in \mathcal{K}^\spadesuit(S,G)$ and for $K \in \mathcal{K}^\heartsuit(S,G)$ via the theory of types.

Corollary 7.4. *Let $\mathbf{S}$ be a maximal F-split torus in $\mathbf{G}$ and let $K \in \mathcal{K}^\spadesuit(S,G)$. We have*

$$\mathfrak{Z}(\mathcal{H}(G,K)) = \prod_{\mathfrak{s}\in\mathfrak{S}(K)} \mathfrak{Z}(\mathcal{H}(G,\rho)) \simeq \prod_{\mathfrak{s}\in\mathfrak{S}(K)} \mathbb{C}[{}^\dagger J_M/J_M]^{W(\rho_M)}$$

where, for $\mathfrak{s} = [M,\sigma]_G$, (J_M, ρ_M) is a supercuspidal type of $\mathfrak{s}_M$, and (J,ρ) is a G-cover of (J_M, ρ_M).

Proof. This is a consequence of Theorem 4.8. Note that we can apply Theorem 4.8 since the assumption $\mathcal{I}_M(\rho_M) \subset \widetilde{J}_M$ is satisfied by Lemma 7.2. □

Now, we describe the center of $\mathcal{H}(G,K)$ for $K \in \mathcal{K}^{\heartsuit}(S,G)$. Let $\mathbf{M}$ be a Levi subgroup of $\mathbf{G}$ that contains $\mathbf{S}$ and let

$$l_M^G : \mathfrak{B}(M) \to \mathfrak{B}(G), \ [M,\sigma]_M \to [M,\sigma]_G.$$

Let $\mathfrak{S}(K\cap M)_{sc} = \{[M,\sigma]_M \in \mathfrak{S}(K\cap M) \mid \sigma \text{ a supercuspidal representation of } M\}$.

Corollary 7.5. *Let $K \in \mathcal{K}^{\heartsuit}(S,G)$. We have*

$$\mathfrak{Z}(\mathcal{H}(G,K)) \simeq \prod_{[M]} \prod_{\mathfrak{s}_M \in \mathfrak{S}(K\cap M)_{sc}} \mathbb{C}[{}^{\dagger}J_M/J_M]^{W(\rho_M)}$$

where $[\mathbf{M}]$ runs through the $\mathbf{G}$-conjugacy classes of F-Levi subgroups of $\mathbf{G}$ and $\mathbf{M}$ is a representative in this conjugacy class that contains $\mathbf{S}$.

Proof. Note that by Lemma 5.3, we have

$$\mathfrak{S}(K) = \bigsqcup_M l_M^G(\mathfrak{S}(K\cap M)_{sc}).$$

Hence the corollary follows from Corollary 7.4. □

References

[1] Petar Bakić, Gordan Savin, *The Gelfand-Graev representation of SO(2n+1) in terms of Hecke algebras*, arXiv:2011.02456.

[2] Bernstein, J and Deligne, P, *Le "center" de Bernstein*, 1-32, Hermann, Paris, Travaux en cours, Representations of reductive groups over a local field, 1984.

[3] Bestvina, Mladen; Savin, Gordan, *Bounded contractions for affine buildings*, Proc. Amer. Math. Soc. 148 (2020), no. 2, 875–883.

[4] Boumasmoud, Reda, *A tale of parahoric–Hecke algebras, Bernstein and Satake homomorphisms*, arXiv:2106.12500v2, 2021, 1–61.

[5] C. J. Bushnell and P. C. Kutzko, *Smooth representations of reductive p-adic groups: structure theory via types*, Proc. London Math. Soc. (3) 77 (1998), no. 3, 582–634.

[6] C. J. Bushnell and P. C. Kutzko, *The admissible dual of GL(n) via compact open subgroups*, Annals of Mathematics Studies 129 (Princeton University Press, 1993).

[7] C. J. Bushnell, *Representations of reductive p-adic groups: localization of Hecke algebras and applications. (English summary)* J. London Math. Soc. (2) 63 (2001), no. 2, 364–386.

[8] Bushnell, Colin J.; Henniart, Guy, *Generalized Whittaker models and the Bernstein center*, Amer. J. Math. 125 (2003), no. 3, 513–547.

[9] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local*, Inst. Hautes Études Sci. Publ. Math. (1972), no. 41, 5–251.

[10] F. Bruhat and J. Tits: *Groupes réductifs sur un corps local:II. Schéma en groupes. Existence d'une donnée radicielle valuée*, Inst. Hautes Études Sci. Publ. Math. 60 (1984), 197–376.

[11] Dat, J.-F., *Types et inductions pour les représentations modulaires des groupes p-adiques. (French. English, French summary) [Types and inductions for modular representations of p-adic groups] With an appendix by Marie-France Vignéras.* Ann. Sci. École Norm. Sup. (4) 32 (1999), no. 1, 1–38.

[12] Fintzen, Jessica, *Types for tame p-adic groups.* Ann. of Math. (2) 193 (2021), no. 1, 303–346.

[13] Fintzen, Jessica, *On the construction of tame supercuspidal representations.* Compos. Math. 157 (2021), no. 12, 2733–2746.

[14] Kim, Ju-Lee ; Yu, Jiu-Kang . *Construction of tame types.* Representation theory, number theory, and invariant theory, 337–357, Progr. Math., 323, Birkhäuser/Springer, Cham, 2017.

[15] Kottwitz Robert Edward, *Isocrystals with additional structure. II*, Compositio Mathematica 109 (1997), no. 3, 255–339. +++

[16] M. Miyauchi and S. Stevens, *Semisimle types for p-adic classical groups*, Mathematische Annalen, 358 (2014), pp. 257–288.

[17] A. Moy and G. Prasad, *Unramified minimal K-types for p-adic groups.* Invent. Math. 116 (1994), 393-408.

[18] A. Moy and G. Prasad, *Jacquet functors and unramified minimal K-types*. Comment. Math. Helvetici 71 (1996), 98-121.

[19] Kazuma Ohara, *Hecke algebras for tame supercuspidal types*, arxiv 2101.01873.

[20] A. Roche, *Types and Hecke algebras for principal series representations of split reductive p-adic groups*, Ann. Sci. École Norm. Sup. (4), Volume 31, 1998, No. 3, 361–413.

[21] ———, *The Bernstein decomposition and the Bernstein center*, Ottawa lectures on admissible representations of reductive p-adic groups, Fields Inst. Monogr., vol. 26, Amer. Math. Soc., Providence, RI, 2009, pp. 3–52. MR2508719

[22] Sean Rostami, *The Bernstein presentation for general connected reductive groups*, Journal of the London Mathematical Society. Second Series, Volume 91, 2015, 514–536.

[23] Robert Steinberg, *Regular elements of semisimple algebraic groups*, Inst. Hautes Études Sci. Publ. Math. (1965), no. 25, 49–80.

[24] Vignéras, Marie-France, *Représentations ℓ-modulaires d'un groupe réductif p-adique avec* $\ell \neq p$, Boston: Birkhäuser, 1996.

[25] Yu, Jiu-Kang . *Construction of tame supercuspidal representations.* J. Amer. Math. Soc. 14 (2001), no. 3, 579–622.